16th International Conference on Information Fusion
Istanbul, Turkey, July 9-12, 2013

DS-Based Uncertain Implication Rules for Inference and Fusion Applications

Rafael C. Núñez*, Ranga Dabarera*, Matthias Scheutz†,
Gordon Briggs†, Otávio Bueno‡, Kamal Premaratne*, Manohar N. Murthi*
* Department of Electrical and Computer Engineering, University of Miami, Coral Gables, Florida 33146, USA
Email: nunez@umiami.edu; {ranga, kamal, mmurthi}@miami.edu
† Department of Computer Science, Tufts University, Medford, Massachusetts 02155, USA
Email: {matthias.scheutz, gordon.briggs}@tufts.edu
‡ Department of Philosophy, University of Miami, Coral Gables, Florida 33124, USA
Email: otaviobueno@me.com

***Abstract*—Numerous applications rely on implication rules either as models of causal relations among data, or as components of their reasoning and inference systems. Although mature and robust models of implication rules already exist for "perfect" (e.g., boolean) scenarios, there is still a need for improving implication rule models when the data (or system models) are uncertain, ambiguous, vague, or incomplete. Decades of research have produced models for probabilistic and fuzzy systems. However, the work on uncertain implication rules under the Dempster-Shafer (DS) theoretical framework can still be improved. Given that DS theory provides increased robustness against uncertain/incomplete data, and that DS models can easily be converted into probabilistic and fuzzy models, a DS-based implication rule that is consistent with classical logic would definitely improve inference methods when dealing with uncertainty. We introduce a DS-based uncertain implication rule that is consistent with classical logic. This model satisfies reflexivity, contrapositivity, and transitivity properties, and is embedded into an uncertain logic reasoning system that is itself consistent with classical logic. When dealing with "perfect" (i.e., no uncertainty) data, the implication rule model renders the classical implication rule results. Furthermore, we introduce an ambiguity measure to track degeneracy of belief models throughout inference processes. We illustrate the use and behavior of both the uncertain implication rule and the ambiguity measure in a human-robot interaction problem.**

I. Introduction

Implication rules, which model elementary statements of the form "if A, then B," are extensively used for reasoning and inference in a variety of areas such as target identification [1], data mining [2], credibility estimation [3], and sensor networks [4]. Being at the backbone of inference engines in numerous applications, robust methods for handling implication rules in "perfect" scenarios (i.e., those scenarios that can be fully characterized without uncertainty) already exist. However, the use of implication rules in scenarios characterized by ambiguous, vague, or incomplete information, is still a matter of study.

The most mature models of uncertain implication rules are based on probabilistic and fuzzy logic theories. More recent approaches have introduced implication rules in the interval-based Dempster-Shafer (DS) theoretic models that can represent upper and lower bounds to probabilistic models, and that can be converted to probability distributions. DS models would benefit from enhanced uncertain implication models that, for example, preserve the characteristics of classical logic implication rules.

A. Uncertain implication rules

One of the most common models of uncertain implication rules is based on Bayesian statistics. In particular, conditional probabilities are often used to model causal relations between data in a probabilistic context [5]. However, it is widely acknowledged that the use of conditional probabilities as models for uncertain implication rules is not always appropriate [5], [6]. Further refinements of probabilistic models, such as the ones presented in [7] and [8], aim at enhancing consistency of the uncertain implications.

Fuzzy implications have also been extensively studied, rendering multiple models such as the Mamdani [9] and Zadeh [10] implications. Some of these models are tuned for particular applications. With a wide variety of implication models, the problem has often translated into selecting the appropriate implication rule for a given application [11]. In addition, fuzzy logic reasoning systems usually demand very specific domain knowledge, as it requires designing the membership functions that better represent the uncertainties in the data and/or models.

As an alternative, DS implication models aim at extending "perfect" and relatively simpler logic reasoning systems with the capability of modeling uncertainties based on intervals. Some early approaches include the implication models introduced in [12] and [13]. Based on the Dempster Combination Rule, these models enable only small fragments of classical logic inferences under the DS framework. They satisfy some basic properties such as reflexivity, transitivity, and contrapositivity. These models, however, do not support a wider logic-consistency analysis that could make them useful to enhance classical logic inference methods with uncertainty measures. In this paper, we propose an approach that aims at enhancing consistency with classical propositional logic operations.

B. Contributions

In this paper we focus on DS-based models for implication rules. We analyze conditions that may lead to degenerate belief models when reasoning with existing (DCR-based) uncertain implication rules. Furthermore, we introduce an ambiguity measure that can track the degeneracy of belief models, giving an indication of the usefulness of the inference result. This ambiguity measure is based on plausibility transformations.

978-605-86311-1-3

In addition, we introduce a classically-preserving implication rule for DS models. This implication rule is based on the Conditional Fusion Equation [14]. The preservation of classical logic is attained by ensuring that the implication model and logic reasoning system satisfy basic properties, inherited from classical logic, such as reflexivity, transitivity, contrapositivity, idempotency, associativity, commutativity, and distributivity. Furthermore, when the uncertainty intervals $[\alpha, \beta]$ that characterize an implication rule and its operands are such that they represent a perfect scenario (i.e., $\alpha = \beta \in \{0, 1\}$), the implication rule renders the corresponding classical logic results.

II. Notation and basic definitions

DS Theory is defined for a discrete set of elementary events related to a given problem. This set is called the *Frame of Discernment* (FoD). In general, a FoD is defined as $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$, and has a finite cardinality $N = |\Theta|$. The power set of Θ is defined as a set containing all the possible subsets of Θ, *i.e.*, $2^\Theta = \{A : A \subseteq \Theta\}$. Next we introduce some basic definitions of DS Theory, as required for explaining models for uncertain implication rules. For details on DS Theory, we refer the reader to [15], [16].

A. Basic Belief Assignment

A Basic Belief Assignment (BBA) or *mass assignment* is a mapping $m_\Theta(\cdot) : 2^\Theta \to [0, 1]$ such that: $\sum_{A\subseteq\Theta} m_\Theta(A) = 1$ and $m_\Theta(\emptyset) = 0$. The BBA measures the support assigned to $A \subseteq \Theta$. Masses in DS theory can be assigned to any singleton or non-singleton proposition. A belief function is called Bayesian if each focal element in Θ is a singleton. The subsets A such that $m(A) > 0$ are referred to as focal elements of the BBA. The set of focal elements is the core $\mathcal{F}_\Theta$. The triple $\{\Theta, \mathcal{F}_\Theta, m_\Theta(\cdot)\}$ is referred to as *Body of Evidence* (BoE).

B. Belief and Plausibility

Given a BoE $\{\Theta, \mathcal{F}, m\}$, the *belief* function $\mathrm{Bel} : 2^\Theta \to [0, 1]$ is defined as: $\mathrm{Bel}_\Theta(A) = \sum_{B\subseteq A} m_\Theta(B)$. $\mathrm{Bel}(A)$ represents the total belief that is committed to A without also being committed to its complement A^C. The *plausibility* function $\mathrm{Pl} : 2^\Theta \to [0, 1]$ is defined as: $\mathrm{Pl}_\Theta(A) = 1 - \mathrm{Bel}_\Theta(A^C)$. It corresponds to the total belief that does not contradict A. The *uncertainty* of A is: $[\mathrm{Bel}_\Theta(A), \mathrm{Pl}_\Theta(A)]$.

C. Combination Rules

Dempster Combination Rule (DCR). For two focal sets $C \subseteq \Theta$ and $D \subseteq \Theta$ such that $B = C \cap D$, and two BBA's $m_j(\cdot)$ and $m_k(\cdot)$, the combined $m_{jk}(B)$ is given by: $m_{jk}(B) = \frac{1}{1-K_{jk}} \sum_{C\cap D=B; B\neq\emptyset} m_j(C)\, m_k(D)$, where $K_{jk} = \sum_{C\cap D=\emptyset} m_j(C)\, m_k(D) \neq 1$ is referred to as the *conflict* between the two BBAs; $K_{jk} = 1$ identifies two totally conflicting BBAs for which DCR-based fusion cannot be carried out.

Conditional Fusion Equation (CFE). A combination rule that is robust when confronted with conflicting evidence is the *Conditional Fusion Equation (CFE)* [17], which is based on the DS theoretic conditional approach [14]. The CFE combines M BBAs as [17]: $m(B) = \sum_{i=1}^{M} \sum_{A_i\in\mathcal{A}_i} \gamma_i(A_i)\, m_i(B|A_i)$, where $\sum_{i=1}^{M} \sum_{A_i\in\mathcal{A}_i} \gamma_i(A_i) = 1$. Here $\mathcal{A}_i = \{A \in \mathcal{F}_i : \mathrm{Bel}_i(A) > 0\}$, $i = 1, \ldots, M$. The conditionals are computed using Fagin-Halperns' Rule of Conditioning [18].

III. DCR-based Uncertain Implications

As mentioned in Section I above, there are a number of approaches for modeling uncertain implications. Most of these models are based on DCR, including the models presented in [1], [12], and [13]. The latter, introduced by Benavoli, et. al., has been shown to satisfy important properties from classical logic. In particular, it satisfies reflexivity, transitivity, and contrapositivity [13]. These properties make this rule one of the most complete DS representations of uncertain implications.

In this section we briefly describe the uncertain implication in [13], as it applies to a model of uncertain logic reasoning. Then, we show some limitations of this model, and introduce a new ambiguity measure of a BBA. This measure can aid in identifying if inferred BBAs provide useful information for making decisions given uncertain evidence.

A. Uncertain Logic Inference based on DCR

Consider the propositions $\varphi(x)$, with uncertainty $[\alpha_x, \beta_x]$, and $\psi(y)$, with uncertainty $[\alpha_y, \beta_y]$. DS models for these propositions can be defined on the FoDs $\Theta_{\varphi,x} = \{\varphi(x) \times \mathbf{1}, \varphi(x) \times \mathbf{0}\}$, and $\Theta_{\psi,y} = \{\psi(y) \times \mathbf{1}, \psi(y) \times \mathbf{0}\}$. The focal element $\varphi(x) \times \mathbf{1}$ represents the event "x satisfies property φ", while the focal element $\varphi(x) \times \mathbf{0}$ represents the event "x does not satisfy property φ". Similarly, the focal element $\psi(y) \times \mathbf{1}$ represents the event "y satisfies property ψ", while the focal element $\psi(y) \times \mathbf{0}$ represents the event "y does not satisfy property ψ". When no confusion can arise, we will represent the elements of $\Theta_{\varphi,x}$ as $\{x, \overline{x}\}$, and the elements of $\Theta_{\psi,y}$ as $\{y, \overline{y}\}$. The BBAs that model these propositions are then:

$$\varphi(x) : m_x(x) = \alpha_x; m_x(\overline{x}) = 1 - \beta_x;\ m_x(\Theta_{\varphi,x}) = \beta_x - \alpha_x; \tag{1}$$

and

$$\psi(y) : m_y(y) = \alpha_y; m_y(\overline{y}) = 1 - \beta_y;\ m_y(\Theta_{\psi,y}) = \beta_y - \alpha_y. \tag{2}$$

Now, consider the uncertain implication rule $\varphi(x) \implies \psi(y)$, with uncertainty $[\alpha_R, \beta_R]$. Benavoli's implication renders, in this case:

$$m_{xy}(C) = \begin{cases} \alpha_R, & \text{if } C = (x \times y) \cup (\overline{x} \times \Theta_{\psi,y}); \\ 1 - \beta_R, & \text{if } C = (\overline{x} \times \overline{y}) \cup (x \times \Theta_{\psi,y}); \\ \beta_R - \alpha_R, & \text{if } C = \Theta_{\varphi,x} \times \Theta_{\psi,y}, \end{cases} \tag{3}$$

defined over the FoD $\Theta_{\varphi,x} \times \Theta_{\psi,y}$. If the uncertainty parameters of the antecedent (i.e., $[\alpha_x, \beta_x]$) and the rule (i.e., $[\alpha_R, \beta_R]$) are known, then it is possible to fuse m_x with m_{xy}, and marginalize into $\Theta_{\psi,y}$ to obtain the uncertainty parameters of the consequent, $[\alpha_y, \beta_y]$. Using DCR these parameters are:

$$\alpha_y = \alpha_x \alpha_R, \quad \text{and} \quad \beta_y = 1 - (1 - \beta_x)(1 - \beta_R). \tag{4}$$

Although (4) can be useful in estimating uncertainty of the consequent and solving some inference problems, it should be used carefully, as there may exist conditions that create problems for this type of inference. For example, the uncertainty of the resulting BBAs could increase if we cascade several implication rules. In particular, consider the cascaded rule model $\varphi_1 \implies \varphi_2; \varphi_2 \implies \varphi_3; \ldots; \varphi_{N-1} \implies \varphi_N$. Also, let $[\alpha_{i,j}, \beta_{i,j}]$ be the uncertainty interval of the rule

$\varphi_i \implies \varphi_j$. If we obtain evidence for the truth of φ_1 defined as $[\alpha_1, \beta_1]$, then it can be shown that the uncertainty interval $[\alpha_N, \beta_N]$ obtained from (3) and (4) is:

$$\alpha_N = \alpha_1 \prod_{i=1}^{N-1} \alpha_{i,i+1}, \text{ and}$$

$$\beta_N = 1 - (1-\beta_1) \prod_{i=1}^{N-1} (1-\beta_{i,i+1}). \quad (5)$$

If $\alpha_{i,i+1} = \alpha, \beta_{i,i+1} = \beta, \forall i, i = 1, 2, \cdots, N-1$, and $0 \leq \alpha < \beta < 1$, (5) becomes:

$$\alpha_N = \alpha_1 \alpha^{N-1}, \quad \text{and} \quad \beta_N = 1 - (1-\beta_1)(1-\beta)^{N-1}.$$

Note that as $N \to \infty$, $[\alpha_N, \beta_N] \to [0, 1]$. In this case, the uncertainty of the consequent is increasing every time we incorporate one new rule. A similar problem occurs if there is at least one value $\alpha_{i,i+1} = 0$, and one value $\beta_{i,i+1} = 1$. We call this event a degeneration of the resulting BBAs. In addition to the cascaded implications, it can also be shown that other DCR-based logic operations could create problems when used for logic inference, e.g., DCR models for AND and OR operations [19].

B. Quantifying the quality of fusion results

To help quantify the degree of degeneracy of BBAs as the DCR-based model above is applied, we introduce an ambiguity measure λ of a BBA; $0 \leq \lambda \leq 1$; with some important characteristics, namely: 1) $\lambda \to 0$ as the BBA degenerates (i.e., as the uncertainty grows); 2) $\lambda \to 1$ as the BBA represents a more exact model (i.e., the BBA gets close to a "perfect" model where $\alpha = \beta \in \{1, 0\}$); and 3) $\lambda \to 0$ as the values of α and $1 - \beta$ get closer to each other.

*Definition 1 (**Ambiguity measure**):* Let $\Theta = \{x_1, x_2, \ldots, x_n\}$ be a FoD, and let m be a BBA defined on Θ. Then, an ambiguity measure λ is defined as:

$$\lambda = 1 + \sum_{x \in \Theta} P_m(x) \log\left(P_m(x)\right), \quad (6)$$

where $P_m(x)$ is the probability of the event x occurring. $P_m(x)$ is obtained from a DS-to-probability transformation applied to the BBA m. ■

Note that this ambiguity measure is similar to the measure introduced in [20]. However, Definition 1 does not rely on the pignistic transformation. This gives us flexibility to select the transformation that renders an ambiguity measure that satisfies the characteristics mentioned above. From several probability transformations available (see, for example, [21], [22], [23], [24], and [25]), we have found that, unlike the others, the plausibility transformation [25] preserves these characteristics. For our dichotomous mass functions (true/false events as in (1) and (2)), the plausibility transformation is defined as:

$$\text{P\_Pl}(x) = \tfrac{1}{\eta}\text{Pl}(x) \quad \text{and} \quad \text{P\_Pl}(\overline{x}) = \tfrac{1}{\eta}\text{Pl}(\overline{x}), \quad (7)$$

with $\eta = \text{Pl}(x) + \text{Pl}(\overline{x})$. Then, the ambiguity measure λ becomes:

$$\lambda = 1 + \text{P\_Pl}(x)\log_2[\text{P\_Pl}(x)] + \text{P\_Pl}(\overline{x})\log_2[\text{P\_Pl}(\overline{x})]. \quad (8)$$

Figure 1 shows how λ changes as a function of the uncertainty parameters $[\alpha, \beta]$ that characterize a dichotomous BBA.

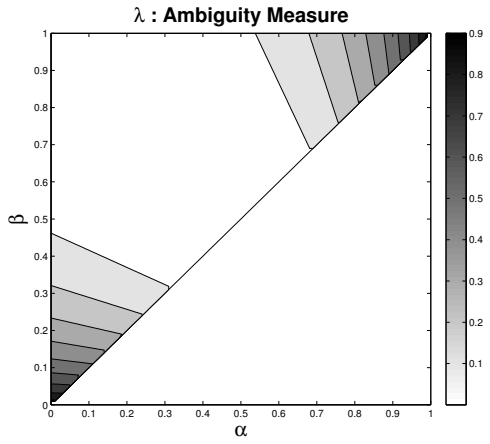

Fig. 1. Ambiguity measure λ as a function of the uncertainty parameters $[\alpha, \beta]$ that characterize a dichotomous BBA. A higher value of λ indicates that the BBA provides stronger evidence of the truth (or falseness) of the focal elements in the corresponding BoE.

IV. CFE-BASED UNCERTAIN IMPLICATIONS

A new method for reasoning with uncertain evidence is introduced in [19]. This framework is consistent with classical logic. This means that it inherits fundamental properties such as idempotency, associativity, commutativity, and distributivity for the AND and OR operators. Furthermore, under no uncertainty (i.e., uncertainty intervals defined by either $[0, 0]$ or $[1, 1]$), operations in this framework converge to those of classical logic. We refer to this framework as *uncertain logic*. In the remainder of this section, we summarize the basic (NOT, AND, OR) operations, and then we elaborate on the uncertain implication rule and its properties. For details and proofs of properties of the basic operations of uncertain logic, we refer the reader to [19].

A. CFE-based uncertain logic

Uncertain logic deals with propositions $(\varphi_1, \varphi_2, \ldots)$ whose truth is uncertain. The level of uncertainty is modeled with DS theory, and is bounded in the range $[0, 1]$. In general, uncertain logic deals with expressions of the form:

$$\varphi(x), \text{ with uncertainty } [\alpha, \beta], \quad (9)$$

where $[\alpha, \beta]$ refers to the corresponding confidence interval, $0 \leq \alpha \leq \beta \leq 1$, $x \in \Theta_x$, and $\Theta_x = \{x_1, x_2, \ldots, x_n\}$. A DS model for expression (9) can be defined over the logical FoD $\Theta_{\varphi,x} = \{\varphi(x) \times \mathbf{1}, \varphi(x) \times \mathbf{0}\}$. This FoD contains two mutually exclusive elements: the extent to which we are confident that the property/proposition φ applies and does not apply to x, respectively. When no confusion can arise, we will represent the elements of $\Theta_{\varphi,x}$ as $\{x, \overline{x}\}$. Using this FoD, a DS model that would capture the information in (9) is:

$$\begin{aligned} \varphi(x): \quad & m(\varphi(x) \times \mathbf{1}) = \alpha; \quad m(\varphi(x) \times \mathbf{0}) = 1 - \beta; \\ & m(\{\varphi(x) \times \mathbf{1}, \varphi(x) \times \mathbf{0}\}) = \beta - \alpha. \end{aligned} \quad (10)$$

Using the alternative notation for the elements in $\Theta_{\varphi,x}$, this is equivalent to:

$$\varphi(x): \quad m(x) = \alpha; \quad m(\overline{x}) = 1 - \beta; \quad m(\Theta_{\varphi,x}) = \beta - \alpha, \quad (11)$$

defined over the FoD $\Theta_{\varphi,x} = \{x, \overline{x}\}$.

If we are interested in modeling the uncertainty of a proposition φ applying to particular elements $x_i \in \Theta_x$, we can extend (9) as:

$$\varphi(x_i), \text{ with uncertainty } [\alpha_i, \beta_i], \tag{12}$$

defined over the FoD $\Theta_{\varphi,x_i} = \{x_i, \overline{x_i}\}$, with $[\alpha_i, \beta_i]$ being the corresponding confidence interval, and $0 \leq \alpha_i \leq \beta_i \leq 1$.

Similarly, if we are interested in modeling the uncertainty of a particular proposition $\varphi_i \in \{\varphi_1, \varphi_2, \ldots, \varphi_M\}$ applying to an element $x \in \Theta_x$, we can extend (9) as:

$$\varphi_i(x), \text{ with uncertainty } [\alpha_i, \beta_i], \tag{13}$$

defined over the FoD $\Theta_{\varphi_i,x} = \{x, \overline{x}\}$, with $[\alpha_i, \beta_i]$ being the corresponding confidence interval, and $0 \leq \alpha_i \leq \beta_i \leq 1$.

In a more general case, we could also be interested in modeling the uncertainty of particular propositions $\varphi_i \in \{\varphi_1, \varphi_2, \ldots, \varphi_M\}$ applying to particular elements $x_j \in \Theta_x$. In this case:

$$\varphi_i(x_j), \text{ with uncertainty } [\alpha_{i,j}, \beta_{i,j}], \tag{14}$$

defined over the FoD $\Theta_{\varphi_i,x_j} = \{\varphi_i(x_j) \times \mathbf{1}, \varphi_i(x_j) \times \mathbf{0}\} = \{x_{i,j}, \overline{x_{i,j}}\}$, with $[\alpha_{i,j}, \beta_{ij}]$ being the corresponding confidence interval, and $0 \leq \alpha_{i,j} \leq \beta_{i,j} \leq 1$.

Based on the FoDs defined above, we can define logic operations such as NOT ($\neg$), AND ($\wedge$), and OR ($\vee$). In the following, we define these basic operations. Note that, whenever possible, we define operations in a simple model (e.g., based on (9) instead of (14)). However, the definitions extend to the more complex cases.

Logical Negation. Given an uncertain proposition $\varphi(x)$ defined as in (9), and its corresponding DS model defined by (11), the logical negation of $\varphi(x)$ is given by:

$$\neg\varphi(x), \text{ with uncertainty } [1-\beta, 1-\alpha], \tag{15}$$

where we have used the complementary BBA [26] corresponding to (11) as the DS theoretic model for $\neg\varphi(x)$, i.e.,

$$\neg\varphi(x): \quad m^c(x) = 1-\beta; \quad m^c(\overline{x}) = \alpha; \quad m^c(\Theta_{\varphi,x}) = \beta - \alpha. \tag{16}$$

Logical AND/OR. Suppose that we have M logic predicates, each providing a statement regarding the truth of x with respect to the proposition $\varphi_i(\cdot)$ as defined by model (13). Then, the corresponding DS models are

$$\varphi_i(x): m_i(x) = \alpha_i; m_i(\overline{x}) = 1-\beta_i; m_i(\Theta_{\varphi_i,x}) = \beta_i - \alpha_i, \tag{17}$$

for $i = 1, 2, \ldots, M$. The DS models for the logical AND and OR of the statements in (17) are:

$$\begin{aligned} &\bigwedge_{i=1}^{M} \varphi_i(x): \quad m(\cdot) = \bigcap_{i=1}^{M} m_i(\cdot); \\ \text{and} \quad &\bigvee_{i=1}^{M} \varphi_i(x): \quad m(\cdot) = \left(\bigcap_{i=1}^{M} m_i^c(\cdot) \right)^c, \end{aligned} \tag{18}$$

respectively, where $\bigcap$ denotes an appropriate fusion operator. In particular, if CFE-based fusion is used, these operators can be tuned to ensure consistency with classical logic. For the case $M = 2$, one way of attaining this consistency is by selecting the CFE coefficients as: $\gamma_1(x) = \gamma_2(x) \equiv \gamma(x); \gamma_1(\overline{x}) = \gamma_2(\overline{x}) \equiv \gamma(\overline{x}); \gamma_1(\Theta) = \gamma_2(\Theta) \equiv \gamma(\Theta)$, where $\gamma(x)$, $\gamma(\overline{x})$, and $\gamma(\Theta)$ are defined as follows.

i. Logical AND: If $\delta_1 + \delta_2 \neq 0$:

$\gamma(x) = \frac{\underline{\alpha}(\beta_1+\beta_2) - \underline{\beta}(\alpha_1+\alpha_2)}{2(\delta_1+\delta_2)}$;

$\gamma(\overline{x}) = \frac{1}{2} - \frac{\underline{\beta}(2-\alpha_1-\alpha_2) - \underline{\alpha}(2-\beta_1-\beta_2)}{2(\delta_1+\delta_2)}$; $\quad \gamma(\Theta) = \frac{\underline{\delta}}{\delta_1+\delta_2}$.

If $\delta_1 + \delta_2 = 0$, i.e., $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$:

$\gamma(x) = \frac{\underline{\alpha} - \gamma(\Theta)(\alpha_1+\alpha_2)}{2}$;

$\gamma(\overline{x}) = \frac{(1-\underline{\alpha}) - \delta(\Theta)(2-\alpha_1-\alpha_2)}{2}$; $\quad \gamma(\Theta) =$ arbitrary.

ii. Logical OR: If $\delta_1 + \delta_2 \neq 0$:

$\gamma(x) = \frac{1}{2} - \frac{\overline{\beta}(2-\alpha_1-\alpha_2) - \overline{\alpha}(2-\beta_1-\beta_2)}{2(\delta_1+\delta_2)}$;

$\gamma(\overline{x}) = \frac{\overline{\alpha}(\beta_1+\beta_2) - \overline{\beta}(\alpha_1+\alpha_2)}{2(\delta_1+\delta_2)}$; $\quad \gamma(\Theta) = \frac{\overline{\delta}}{\delta_1+\delta_2}$.

If $\delta_1 + \delta_2 = 0$, i.e., $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$:

$\gamma(x) = \frac{\overline{\alpha} - \gamma(\Theta)(\alpha_1+\alpha_2)}{2}$;

$\gamma(\overline{x}) = \frac{(1-\overline{\alpha}) - \delta(\Theta)(2-\alpha_1-\alpha_2)}{2}$; $\quad \gamma(\Theta) =$ arbitrary.

In the equations above: $\underline{\alpha} = \min(\alpha_1, \alpha_2)$; $\underline{\beta} = \min(\beta_1, \beta_2)$; $\overline{\alpha} = \max(\alpha_1, \alpha_2)$; $\overline{\beta} = \max(\beta_1, \beta_2)$; $\delta_1 = \beta_1 - \alpha_1$; $\delta_2 = \beta_2 - \alpha_2$; $\underline{\delta} = \underline{\beta} - \underline{\alpha}$; and $\overline{\delta} = \overline{\beta} - \overline{\alpha}$.

When used for the AND operation, this selection of coefficients renders:

$$\begin{aligned} \varphi_1(x) \wedge \varphi_2(x): \quad & m(x) = \underline{\alpha}; \; m(\overline{x}) = 1 - \underline{\beta}; \text{ and} \\ & m(\Theta_{\varphi_1,x} \times \Theta_{\varphi_2,x}) = \underline{\beta} - \underline{\alpha}. \end{aligned} \tag{19}$$

When used for the OR operation, this selection of coefficients renders:

$$\begin{aligned} \varphi_1(x) \vee \varphi_2(x): \quad & m(x) = \overline{\alpha}; \; m(\overline{x}) = 1 - \overline{\beta}; \text{ and} \\ & m(\Theta_{\varphi_1,x} \times \Theta_{\varphi_2,x}) = \overline{\beta} - \overline{\alpha}. \end{aligned} \tag{20}$$

B. CFE-based implication rules

An uncertain logic implication that is consistent with classical logic can be defined by extending the (classical) definition for the implication rule based on AND/OR operators to the uncertain logic framework. The classical logic definition is: Given two statements $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$, an implication rule in propositional logic has the property:

$$\begin{aligned} \varphi_1(x) \implies \varphi_2(y) &\equiv \neg\varphi_1(x) \vee \varphi_2(y) \\ &\equiv \neg\left(\varphi_1(x) \wedge \neg\varphi_2(y)\right), \end{aligned}$$

where $x_i \in \Theta_x$ and $y_j \in \Theta_y$. Now, consider the case where the antecedent $\varphi_1(x)$ and/or the consequent $\varphi_2(y)$ are/is uncertain, with uncertainty intervals $[\alpha_{\varphi_1,x}, \beta_{\varphi_1,x}]$ and $[\alpha_{\varphi_2,y}, \beta_{\varphi_2,y}]$, respectively. Furthermore, suppose that said uncertainty is represented via the DS theoretic models $m_1(\cdot)$ and $m_2(\cdot)$ over the logical FoDs $\Theta_{\varphi_1,x}$ and $\Theta_{\varphi_2,y}$, respectively (as in (13)). Then, the implication rule $\varphi_1(\cdot) \implies \varphi_2(\cdot)$ is taken to have the following DS model:

$$m_{\varphi_1 \to \varphi_2}(\cdot) = (m_1^c \vee m_2)(\cdot) = (m_1 \wedge m_2^c)^c(\cdot), \tag{21}$$

over the FoD $\Theta_{\varphi_1,x} \times \Theta_{\varphi_2,y}$. This model renders the BBA:

$$\begin{aligned} \varphi_1(x) \implies \varphi_2(y): & \\ m_{\varphi_1\to\varphi_2}(x \times \Theta_{\varphi_2,y}) &= \tfrac{1}{2}\alpha_R; \\ m_{\varphi_1\to\varphi_2}(\Theta_{\varphi_1,x} \times y) &= \tfrac{1}{2}\alpha_R; \\ m_{\varphi_1\to\varphi_2}(\overline{x} \times \Theta_{\varphi_2,y}) &= \tfrac{1}{2}(1-\beta_R); \\ m_{\varphi_1\to\varphi_2}(\Theta_{\varphi_1,x} \times \overline{y}) &= \tfrac{1}{2}(1-\beta_R); \\ m_{\varphi_1\to\varphi_2}(\Theta_{\varphi_1,x} \times \Theta_{\varphi_2,y}) &= \beta_R - \alpha_R, \end{aligned} \tag{22}$$

with $\alpha_R = \max(1-\beta_1, \alpha_2)$ and $\beta_R = \max(1-\alpha_1, \beta_2)$. Note that α_R and β_R define the uncertainty interval $[\alpha_R, \beta_R]$ of the implication rule. This interval is obtained from projecting the BBA defined in (22) into the true-false BoE $\{\mathbf{1}, \mathbf{0}\}$, i.e.,:

$$\begin{aligned} m_{\varphi_1\to\varphi_2}(\mathbf{1}) &= m_{\varphi_1\to\varphi_2}(x \times \Theta_{\varphi_2,y}) \\ &\quad + m_{\varphi_1\to\varphi_2}(\Theta_{\varphi_1,x} \times y) \quad = \alpha_R; \\ m_{\varphi_1\to\varphi_2}(\mathbf{0}) &= m_{\varphi_1\to\varphi_2}(\overline{x} \times \Theta_{\varphi_2,y}) \\ &\quad + m_{\varphi_1\to\varphi_2}(\Theta_{\varphi_1,x} \times \overline{y}) \quad = 1-\beta_R; \\ m_{\varphi_1\to\varphi_2}(\{\mathbf{1}, \mathbf{0}\}) &= m_{\varphi_1\to\varphi_2}(\Theta_{\varphi_1,x} \times \Theta_{\varphi_2,y}) = \beta_R - \alpha_R. \end{aligned} \tag{23}$$

The DS models for implication rules defined by (21) provide us with an important inference tool. For example, given two models $m_1(\cdot)$ and $m_2(\cdot)$ we could model the uncertainty of an implication rule $\varphi_1(x) \implies \varphi_2(y)$. Also, if we know a model $m_{\varphi_1\to\varphi_2}(\cdot)$ for the implication rule, but we do not know one of the arguments (*e.g.*, we do not know $m_2(\cdot)$), we could obtain models for the unknown argument. In this case:

$$\alpha_2 = \begin{cases} \alpha_R, & \text{if } \alpha_R > 1-\beta_1; \\ [0, \alpha_R], & \text{if } \alpha_R = 1-\beta_1; \text{ and} \\ \text{no solution}, & \text{otherwise}, \end{cases} \tag{24}$$

and

$$\beta_2 = \begin{cases} \beta_R, & \text{if } \beta_R > 1-\alpha_1; \\ [0, \beta_R], & \text{if } \beta_R = 1-\alpha_1; \text{ and} \\ \text{no solution}, & \text{otherwise}. \end{cases} \tag{25}$$

Note that, unlike the DCR-based uncertain implication described in Section III, if there is not enough evidence in $m_1(\cdot)$ to support a conclusion on $m_2(\cdot)$, then there is no solution for $[\alpha_2, \beta_2]$ in the CFE-based implication described above. Then, a metric such as the ambiguity measure (λ) introduced in Section III becomes less necessary. Other important observations regarding the CFE-based implication are:

1) In the perfect case (i.e., $\alpha_1 = \beta_1 \in \{\mathbf{1}, \mathbf{0}\}$ and $\alpha_2 = \beta_2 \in \{\mathbf{1}, \mathbf{0}\}$), the CFE-based uncertain implication rule converges to the conventional logic result (see Table I).
2) Given a pair of uncertainty intervals $[\alpha_1, \beta_1]$ and $[\alpha_R, \beta_R]$, it is not always possible to infer anything about a model for the consequent (i.e., $m_2(\cdot)$). This is consistent with the application of the Modus Ponens (MP) rule, according to which, if we know $\varphi_1(x) \implies \varphi_2(y)$, and also that $\varphi_1(x)$ is true, then we can infer that $\varphi_2(y)$ is true. However, if we know that $\varphi_1(x)$ is false (i.e., $\neg\varphi_1(x)$ is true), we cannot say anything about $\varphi_2(y)$.
3) Furthermore, once we have enough support in $m_1(\cdot)$ to infer something regarding $m_2(\cdot)$, the only conclusion that we can provide is that $m_2(\cdot)$ is as uncertain as $m_R(\cdot)$. That is, after we have gathered a certain amount of evidence regarding the truth of the antecedent, getting more evidence is not going to affect the confidence we have on the truth of the consequent (α_2 is bounded by α_R, and β_2 is bounded by β_R).
4) When $\alpha_R = 1-\beta_1$ and $\beta_R = 1-\alpha_1$, an infinite number of solutions exist for $[\alpha_2, \beta_2]$. In this case, we could use the minimum commitment criterion to decide $\alpha_2 = 0$ and $\beta_2 = \beta_R$.

TABLE I. CFE-BASED IMPLICATION: UNCERTAINTY PARAMETERS ARE DEFINED SO THAT THEY REPRESENT COMPLETE CERTAINTY ON THE TRUTH (OR FALSITY) OF EACH PROPOSITION. IN THIS CASE, THERE IS COMPLETE CERTAINTY OF THE TRUTH OF THE OUTPUT MODEL (AS OCCURS WITH CLASSICAL LOGIC.)

Parameters		Uncertainty of the rule
$[\alpha_1, \beta_1]$	$[\alpha_2, \beta_2]$	$[\alpha_R, \beta_R]$
$[0, 0]$	$[0, 0]$	$[1, 1]$
$[0, 0]$	$[1, 1]$	$[1, 1]$
$[1, 1]$	$[0, 0]$	$[0, 0]$
$[1, 1]$	$[1, 1]$	$[1, 1]$

C. Semantics

In classical logic there are two truth-values, "true" and "false". An expression that is true for all interpretations is called a tautology ("$\top$"). An expression that is not true for any interpretation is a contradiction ("$\bot$"). Two expressions are semantically equivalent if they take on the same truth value for all interpretations.

In uncertain logic we extend these definitions. The truth value of an expression corresponds to the support that is projected into the true-false FoD, $\Theta_{t-f} = \{\mathbf{1}, \mathbf{0}\}$. A BBA (10) defined by $[\alpha, \beta] = [1, 1]$ corresponds to the classical logical truth. A BBA (10) defined by $[\alpha, \beta] = [0, 0]$ corresponds to the classical logical falsehood.

The notions of tautology and contradiction in uncertain logic are extended following an approach similar to that in [27]. In particular, given a generic dichotomous BBA ψ characterized by the uncertainty interval $\sigma = [\alpha, \beta]$, we define a σ-tautology as $\top_\sigma \equiv \psi \vee \neg\psi$, and a σ-contradiction as $\bot_\sigma \equiv \psi \wedge \neg\psi$. It follows that $\top \equiv \top_{\sigma=[1,1]}$, and $\bot \equiv \bot_{\sigma=[0,0]}$.

D. Properties of the implication rule

The uncertain implication rule defined in subsection B satisfies the reflexivity, contrapositivity, and transitivity properties:

1) *Reflexivitiy.* Consider a proposition (9) (i.e., $\varphi(x)$, with uncertainty $[\alpha, \beta]$), and its corresponding DS model (10). Then $\varphi(x) \implies \varphi(x)$. Note that, from (21) $\varphi(x) \implies \varphi(x) \equiv \neg\varphi(x) \vee \varphi(x) \equiv \top_{\sigma=[\alpha,\beta]}$.
2) *Contrapositivity.* Consider two propositions $\varphi_1(x)$ and $\varphi_2(x)$, with uncertainty intervals defined as in (13). Then $(\varphi_1(x) \implies \varphi_2(x)) \implies (\neg\varphi_2(x) \implies \neg\varphi_1(x))$. Note that, from (21) $\varphi_1(x) \implies \varphi_2(x) \equiv \neg\varphi_1(x) \vee \varphi_2(x) \triangleq \psi(x)$. Also, from (21) $\neg\varphi_2(x) \implies \neg\varphi_1(x) \equiv \varphi_2(x) \vee \neg\varphi_1(x) \equiv \psi(x)$. Then, $(\varphi_1(x) \implies \varphi_2(x)) \implies (\neg\varphi_2(x) \implies \neg\varphi_1(x)) \equiv \psi(x) \implies \psi(x)$, which is a σ-tautology due to the reflexivity property.
3) *Transitivity.* Consider three propositions $\varphi_1(x)$, $\varphi_2(x)$, and $\varphi_3(x)$, with uncertainty intervals defined as in (13). Consider also the implication rules $\varphi_1(x) \implies \varphi_2(x)$, $\varphi_2(x) \implies \varphi_3(x)$, and $\varphi_1(x) \implies \varphi_3(x)$, with uncertainty intervals $[\alpha_{R1}, \beta_{R1}]$, $[\alpha_{R2}, \beta_{R2}]$, and $[\alpha_{R3}, \beta_{R3}]$, respectively. Then, *(i)* assume $\varphi_1(x)$ is true (in classical logic this means: $\alpha_1 = \beta_1 = 1$; in general, this is: $\alpha_{R1} > 1-\beta_1$ and $\beta_{R1} > 1-\alpha_1$, which are the conditions for obtaining a solution $[\alpha_2, \beta_2] = [\alpha_{R1}, \beta_{R1}]$ based on (24) and (25)); *(ii)* $\varphi_2(x)$, with uncertainty $[\alpha_2, \beta_2] = [\alpha_{R1}, \beta_{R1}]$ is obtained from MP of $\varphi_1(x) \implies \varphi_2(x)$ and $\varphi_1(x)$; *(iii)* $\varphi_3(x)$, with uncertainty $[\alpha_3, \beta_3] =$

$[\alpha_{R2}, \beta_{R2}]$ is obtained from MP of $\varphi_2(x) \implies \varphi_3(x)$ and *(ii)*; *(iv)* by conditional introduction (*(i)* and (*iii*)) we obtain $\varphi_1(x) \implies \varphi_3(x)$, with uncertainty $[\alpha_{R3}, \beta_{R3}] = [\max(1-\beta_1, \alpha_3), \max(1-\alpha_1, \beta_3)]$. Note that, in the general case, $\alpha_3 \geq \alpha_1$, and $\beta_3 \geq \beta_1$, which is consistent with the implication rule model in classical logic.

E. Rules of Inference

Inference in uncertain logic shares the fundamental principles of classical logic, and adds the possibility of attaching, tracking, and propagating uncertainties that may arise on premises and/or rules. One of the basic rules of inference is Modus Ponens (MP). An uncertain logic model for MP is provided by (24) and (25). Similarly, as described in [19], it is possible to obtain uncertain logic models for other rules of inference, such as Modus Tollens (MT), AND elimination (AE), AND introduction (AI), universal instantiation (UI), and existential instantiation (EI).

V. Case study: Human-robot interaction

In this section we illustrate, through an example, how the ambiguity measure and the CFE-based implication rule can be used in a practical application. In particular, we show an application on human-robot interaction, as this type of application allows the inference engine to actively request more information/data for refining conclusions when existing evidence does not lead to a final conclusion.

Consider a human H giving an implicit instruction to robot R with uncertainty boundaries $[\alpha_1, \beta_1]$. This interval reflects the degree to which R believes that H's statement is true. Note that this uncertainty may change depending on the parsing process, as well as on the actual instruction provided by H. For example, in a real-life scenario, H may use words (or indicate rules to R) that entail uncertainty, such as *usually*, *typically*, or *generally*.

In this example, the instruction provided by H is: "Commander Z really needs a medkit". The robot then runs an inference process in which R needs to find if it needs to get the medkit for Z or not. If the conclusion is very precise (i.e., low uncertainty), then R could simply execute the required action, which could be either get the medkit for Z or not. However, if the conclusion is highly ambiguous, then the robot R could respond "Should I get it for him?" and solve the ambiguity problem.

An inference process to solve R's problem could be as follows. Suppose that the following rules were given (in natural language) to R:

1) "if *x* needs *y*, then *x* has a goal to have *y*", with uncertainty $[\alpha_2, \beta_2]$;
2) "Commander Z is likely of higher rank than robot R", with uncertainty $[\alpha_3, \beta_3]$;
3) "*usually*, if *x* is of higher rank than *y* and *x* has goal *g*, then *y* should have the goal for *x* to have goal *g*, with uncertainty $[\alpha_4, \beta_4]$; and
4) "if the robot has a goal for *x* to have a goal to have *y*, then the robot *should have* the goal to get *y* for *x*", with uncertainty $[\alpha_5, \beta_5]$.

TABLE II. Inference process for the human-robot interaction problem of the case study (M = medkit).

	Logic Formula	Premises	Uncert.
1	$\texttt{Needs}(\text{Z}, \text{M})$	Δ	$[\alpha_1, \beta_1]$
2	$\forall x\ \forall y:\ \texttt{Needs}(x, y) \implies \texttt{Goal}(x, \texttt{Have}(x, y))$	Δ	$[\alpha_2, \beta_2]$
3	$\texttt{Rank}(\text{Z}) > \texttt{Rank}(\text{R})$	Δ	$[\alpha_3, \beta_3]$
4	$\forall x\ \forall y\ \forall g:\ \texttt{Rank}(x) > \texttt{Rank}(y)) \wedge \texttt{Goal}(x, g) \implies \texttt{Goal}(y, \texttt{Goal}(x, g)$	Δ	$[\alpha_4, \beta_4]$
5	$\forall x\ \forall y:\ \texttt{Goal}(\text{R}, \texttt{Goal}(x, \texttt{Have}(x, y))) \implies \texttt{GetFor}(\text{R}, y, x)$	Δ	$[\alpha_5, \beta_5]$
6	$\texttt{Goal}(\text{Z}, \texttt{Have}(\text{Z}, \text{M}))$	1, 2, MP	$[\alpha_6, \beta_6]$
7	$\texttt{Rank}(\text{Z}) > \texttt{Rank}(\text{R}) \wedge \texttt{Goal}(\text{Z}, \texttt{Have}(\text{Z}, \text{M}))$	3, 6, AI	$[\alpha_7, \beta_7]$
8	$\texttt{Goal}(\text{R}, \texttt{Goal}(\text{Z}, \texttt{Have}(\text{Z}, \text{M})))$	4, 7, MP	$[\alpha_8, \beta_8]$
9	$\texttt{GetFor}(\text{R}, \text{M}, \text{Z})$	5, 8, MP	$[\alpha_9, \beta_9]$

Note that all these expressions entail some uncertainty, which may be due uncertain information already known to the robot, or by imprecise words that were used to describe the instructions, such as "likely", "usually", and "should have".

These instructions can be expressed in first-order logic as is shown in rows 1 to 5 of Table II. These rows represent the premises of our inference process. Based on these premises, an inference process could continue as shown in rows 6 to 9 of Table II. For simplicity, we assume that the required inference rules carry over from classical logic to the uncertainty case (we refer the reader to [19] for details on inference rules and quantifiers in the DS uncertain logic model used in this example).

In order to better understand how the uncertainty propagates in this example, we analyze four cases: A) Perfect scenario (i.e., no uncertainty); B) Probabilistic scenario (i.e., $\alpha_i = \beta_i, i = 1, 2, \ldots, 5$); C) Probabilistic scenario with insufficient evidence; and D) General scenario.

A. Perfect scenario (i.e., no uncertainty)

Figure 2 illustrates a scenario where all the premises and rules are taken as truth. This is, the uncertainty intervals $[\alpha_i, \beta_i], i = 1, 2, \ldots, 5$, for the premises A1 to A5 is $[1, 1]$. In this case, the uncertainty of every step in our reasoning process is defined by the uncertainty interval $[1, 1]$, which is consistent with classical logic results. Note that the result is the same for both of the models analyzed, namely, the CFE-based inference model (top), and the DCR-based inference model (center). The figure also shows, at the bottom, the output of our ambiguity measure λ, which, as expected, remains at 1 throughout all the inference process.

B. Probabilistic scenario

Figure 3 illustrates a scenario where all the premises and rules are probabilistic. That is, the uncertainty intervals $[\alpha_i, \beta_i], i = 1, 2, \ldots, 5$, for the premises A1 to A5 are characterized by $\alpha_i = \beta_i$. In this case, the CFE-based model (top) maintains the probabilistic behavior throughout the inference process. The DCR-based model (center), on the contrary, departs from the probabilistic model and, in this case, the uncertainty increases as the inference process progresses. This is also seen in the ambiguity measure (bottom), as it is always decreasing for the DCR-based inference (becoming more ambiguous).

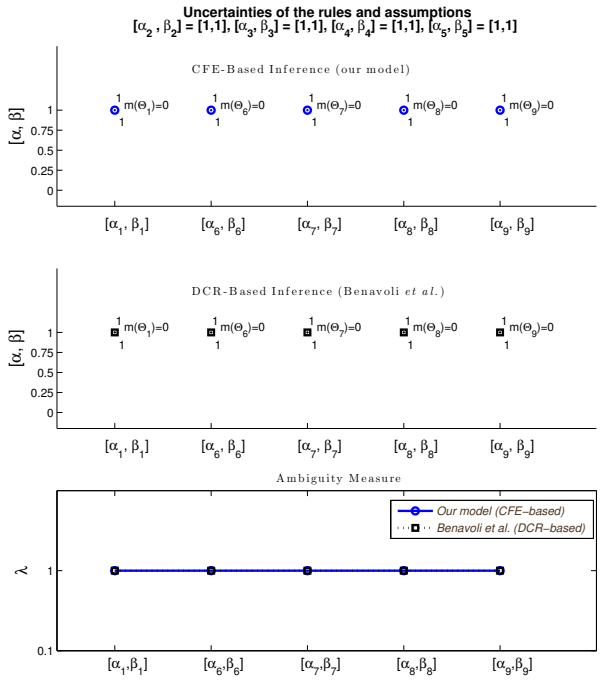

Fig. 2. DS-based uncertain logic inference on a perfect (i.e., no uncertainty) scenario. In this case, there is no uncertainty on any of the inferred premises.

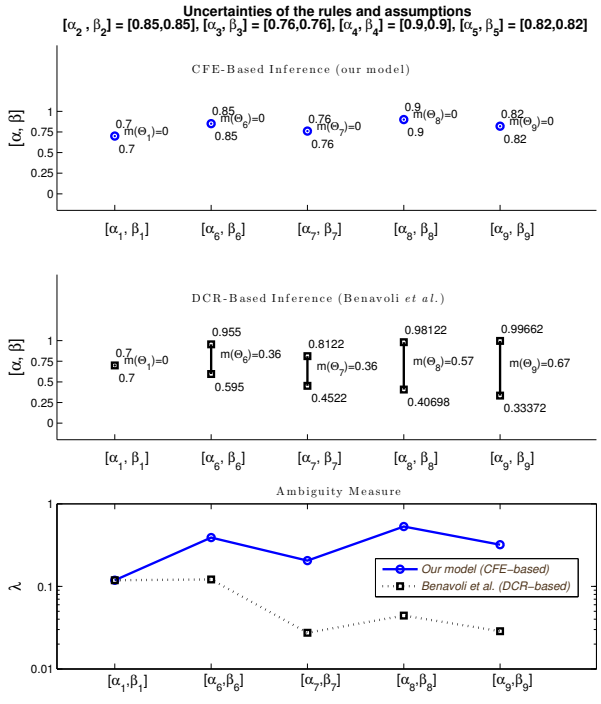

Fig. 3. DS-based uncertain logic inference on a probabilistic scenario. In this case, CFE-based inference maintains the probabilistic behavior throughout the inference process, while DCR-based inference cannot maintain it. The uncertainty of the DCR-based inference increases as the inference process progresses.

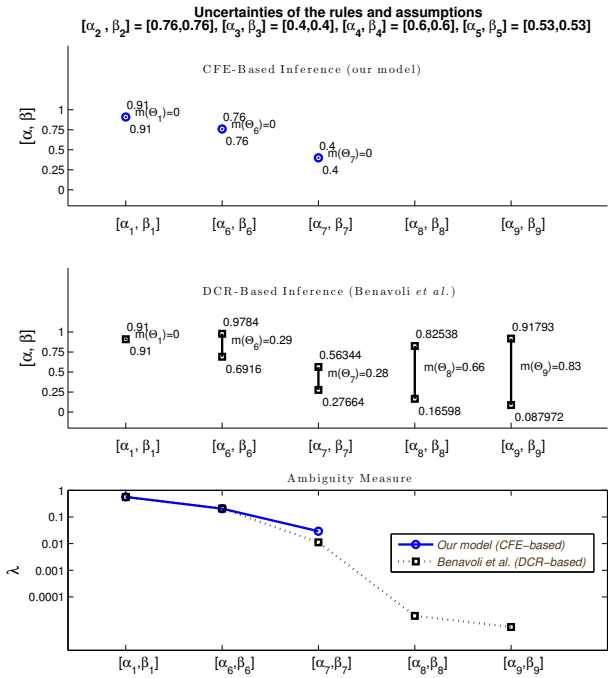

Fig. 4. DS-based uncertain logic inference on a probabilistic scenario with insufficient evidence. In this case, CFE-based inference stops providing results when there is not enough evidence to support them. DCR-based inference, on the contrary, keeps providing results without any indication about the risk of making a decision based on these results.

C. Probabilistic scenario with insufficient evidence

Figure 4 illustrates a scenario where all the premises and rules are probabilistic, but in which the support for the premises makes the inference process stop after a certain number of steps. Note that, as expected, CFE-based inference (top) stops rendering results when the evidence is not enough to provide a conclusion. DCR-based inference (center), on the contrary, keeps delivering results without indicating the risk of making a decision based on the output results. Thus, the ambiguity measure could be incorporated into decision-making processes.

D. General-case scenario

Figure 5 illustrates a general-case scenario where there are no restrictions regarding the uncertainty intervals $[\alpha_i, \beta_i], i = 1, 2, \ldots, 5$, for the premises A1 to A5. In this case, the CFE-based model (top) renders more "certain" results than the DCR-based model (center), as evidenced from the ambiguity measure (bottom).

VI. Conclusion

We introduced a CFE-based uncertain implication rule for reasoning and inference in the DS framework. This implication rule has the ability of capturing the indefiniteness involved in the data, as well as in the knowledge models (e.g., language). It is also consistent with classical logic, rendering the conventional implication results when the scenario represents "perfect" (i.e., without uncertainty) data/models. In addition, we

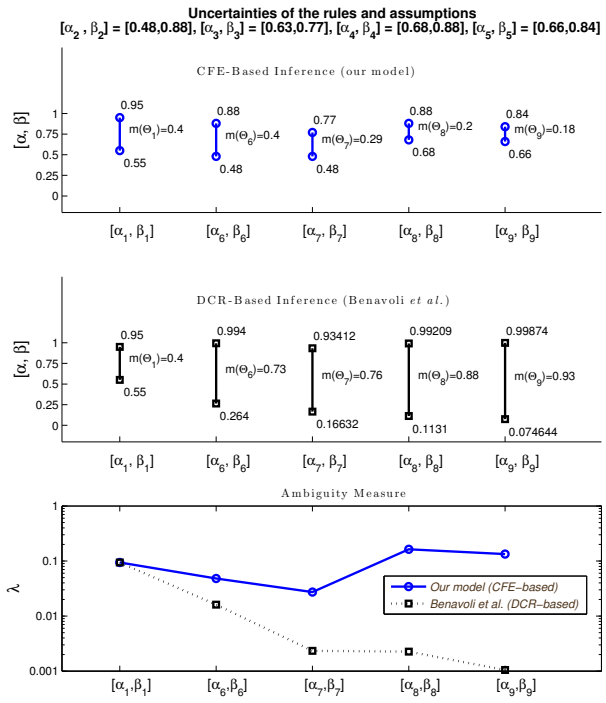

Fig. 5. DS-based inference on a general-case scenario. In this case, although both CFE and DCR-based inference provide uncertain results, the ambiguity measure on the resulting BBAs is higher for the CFE-based inference, as indicated by λ.

introduced an ambiguity measure for tracking degeneracy of belief models throughout the inference process. This measure can indicate if the uncertainty in a particular belief model has grown beyond a threshold that makes the inference result either unreliable or not conclusive.

Acknowledgment

This work is based on research supported by the US Office of Naval Research (ONR) via grants #N00014-10-1-0140. and #N00014-11-1-0493, and the US National Science Foundation (NSF) via grant #1038257.

References

[1] B. Ristic and P. Smets, "Target identification using belief functions and implication rules," *IEEE transactions on Aerospace and Electronic Systems*, vol. 41, July 2005.

[2] M. J. Zaki, "Scalable algorithms for association mining," *IEEE Transactions on Knowledge and Data Engineering*, vol. 12, no. 3, pp. 372–390, 2000.

[3] R. C. Núñez, T. Wickramarathne, K. Premaratne, M. N. Murthi, S. Kübler, M. Scheutz, and M. Pravia, "Credibility assessment and inference for fusion of hard and soft information," in *Proc. 2nd International Conference on Cross-Cultural Decision Making: Focus 2012 (also in Advances in Design for Cross-Cultural Activities, 1, pp. 96-105, Eds: D. D. Schmorrow, D. M. Nicholson, CRC Press, 2013)*, July 2012.

[4] A. Boukerche and S. Samarah, "A novel algorithm for mining association rules in wireless ad hoc sensor networks," *IEEE Transactions on Parallel and Distributed Systems*, vol. 19, no. 7, pp. 865–877, 2008.

[5] M. R. Genesereth and N. J. Nilsson, *Logical foundations of artificial intelligence*. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc., 1987.

[6] D. Lewis, "Probabilities of conditionals and conditional probabilities," *The Philosophical Review*, vol. 85, no. 3, pp. pp. 297–315, 1976.

[7] N. J. Nilsson, "Probabilistic logic," *Artificial Intelligence*, vol. 28, no. 1, pp. 71–88, February 1986.

[8] J. Wang, J. Byrnes, M. Valtorta, and M. Huhns, "On the combination of logical and probabilistic models for information analysis," *Applied Intelligence*, vol. 36, no. 2, pp. 472–497, Mar. 2012.

[9] E. H. Mamdani and S. Assilian, "An experiment in linguistic synthesis with a fuzzy logic controller," *International Journal of Human-Computer Studies*, vol. 51, no. 2, pp. 135 – 147, 1999.

[10] L. Zedeh, "Knowledge representation in fuzzy logic," *IEEE Transactions on Knowledge and Data Engineering*, vol. 1, no. 1, pp. 89–100, 1989.

[11] X. Zhu, Z. Huang, S. Yang, and G. Shen, "Fuzzy implication methods in fuzzy logic," in *Proceedings of the Fourth International Conference on Fuzzy Systems and Knowledge Discovery - Volume 01*, ser. FSKD '07. Washington, DC, USA: IEEE Computer Society, 2007, pp. 154–158.

[12] R. Almond, *Graphical Belief Modeling*. Chapman & Hall, 1995.

[13] A. Benavoli, L. Chisci, A. Farina, and B. Ristic, "Modelling uncertain implication rules in evidence theory," in *11th International Conference on Information Fusion*, 2008.

[14] K. Premaratne, M. N. Murthi, J. Zhang, M. Scheutz, and P. H. Bauer, "A Dempster-Shafer theoretic conditional approach to evidence updating for fusion of hard and soft data," in *12th International Conference on Information Fusion, 2009. FUSION '09*, July 2009, pp. 2122 –2129.

[15] J. Pearl, *Probabilistic reasoning in intelligent systems: networks of plausible inference*. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc., 1988.

[16] G. Shafer, *A mathematical theory of evidence*. Princeton University Press, 1976.

[17] T. Wickramarathne, K. Premaratne, and M. N. Murthi, "Consensus-Based Credibility Estimation of Soft Evidence for Robust Data Fusion," in *Proceedings of the 2nd International Conference on Belief Functions, Compiègne, France, 9-11 May 2012*, ser. Advances in Soft Computing, vol. 164. Springer, 2012, pp. 301–309.

[18] R. Fagin and J. Y. Halpern, "A new approach to updating beliefs," in *Uncertainty in Artificial Intelligence*. Elsevier Science Publishers, 1991, pp. 347–374.

[19] R. C. Núñez, M. Scheutz, K. Premaratne, and M. N. Murthi, "Modeling Uncertainty in First-Order Logic: A Dempster-Shafer Theoretic Approach," in *8th International Symposium on Imprecise Probability: Theories and Applications*, Compiègne, France, July 2013.

[20] A.-L. Jousselme, L. Chunsheng, D. Grenier, and E. Boss, "Measuring ambiguity in the evidence theory," *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans*, vol. 36, no. 5, pp. 890–903, September 2006.

[21] P. Smets, "Decision making in the TBM: the necessity of the pignistic transformation," *International Journal of Approximate Reasoning*, vol. 38, no. 2, pp. 133 – 147, 2005.

[22] P. Smets and R. Kennes, "The transferable belief model," *Artificial Intelligence*, vol. 66, no. 2, pp. 191 – 234, 1994.

[23] M. Daniel, "Probabilistic transformations of belief functions," in *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, ser. Lecture Notes in Computer Science, L. Godo, Ed. Springer Berlin Heidelberg, 2005, vol. 3571, pp. 539–551.

[24] P. Baroni and P. Vicig, "Transformations from imprecise to precise probabilities," in *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, ser. Lecture Notes in Computer Science, T. Nielsen and N. Zhang, Eds. Springer Berlin Heidelberg, 2003, vol. 2711, pp. 37–49.

[25] B. Cobb and P. Shenoy, "A comparison of methods for transforming belief function models to probability models," in *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, ser. Lecture Notes in Computer Science, T. Nielsen and N. Zhang, Eds. Springer Berlin Heidelberg, 2003, vol. 2711, pp. 255–266.

[26] D. Dubois and H. Prade, "A set-theoretic view of belief functions," in *Classic Works of the Dempster-Shafer Theory of Belief Functions*, ser. Studies in Fuzziness and Soft Computing, R. Yager and L. Liu, Eds. Springer Berlin / Heidelberg, 2008, vol. 219, pp. 375–410.

[27] G.-J. Wang and Y. Leung, "Integrated semantics and logic metric spaces," *Fuzzy Sets Syst.*, vol. 136, no. 1, pp. 71–91, May 2003.