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Modeling Uncertainty in First-Order Logic: A Dempster-Shafer Theoretic Approach

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Abstract

First order logic lies at the core of many methods in mathematics, philosophy, linguistics, and computer science. Although important efforts have been made to extend first order logic to the task of handling uncertainty, there is still a lack of a consistent and unified approach, especially within the Dempster-Shafer (DS) theory framework. In this work we introduce a systematic approach for building belief assignments based on first order logic formulas. Furthermore, we outline the foundations of *Uncertain Logic*, a robust framework for inference and modeling when information is available in the form of first order logic formulas subject to uncertainty. Applications include data fusion, rule mining, credibility estimation, and crowd sourcing, among many others.

Keywords. Uncertain Logic, Uncertain Reasoning, Probabilistic Logic, Dempster-Shafer Theory, Belief Theory.

1 Introduction

Natural language processing, artificial intelligence, and graph analysis are among a number of applications that heavily rely on first order logic formulations. Due to its capability for representing knowledge for inference systems, first order logic has been gradually enriched to handle imperfections in real-life data. Some approaches include fuzzy logic and probabilistic logic [1]. These solutions, however, are not well suited for handling scenarios characterized by ranges of uncertainty, or that require modeling evidence in a very strict manner to minimize the risk of inference results leading to wrong conclusions.

Dempster-Shafer theory [2] provides an ideal modeling tool to address this problem. However, although significant effort has been dedicated to modeling uncertainty in logic under DS theory, there is still a need for a unified approach that is consistent with basic logic operations and that provides the support for handling variables and quantifiers. To address this problem we introduce *Uncertain Logic*, which is the extension of first order logic into DS theory. Consider, for example, an expression of the form: $\exists x:\ \varphi(x)$, with uncertainty $[\alpha, \beta]$, where $\varphi(x)$ is a logic predicate that depends on the variable x. The uncertain logic framework allows us to model this sentence, and to combine it with similar ones in order to solve various inference problems. When $\alpha = \beta$, uncertain logic renders probabilistic results. When $\alpha = \beta \in \{0, 1\}$, uncertain logic converges to first order logic. Unlike existing DS models for logic that, in general, cannot guarantee logic consistency for a plurality of logic constructs, uncertain logic preserves this consistency, and can grow to incorporate logic rules and properties without loss of uncertainty measures. By preserving this consistency, it is possible to seamlessly move between the logic and DS domains, and to incorporate both the strength of first order logic for information representation, inference, and resolution, and the strength of DS for representing and manipulating uncertainty in the data.

1.1 Existing Methods for Handling Uncertainty in Logic

The need for reasoning under the presence of uncertainty has lead to important work aimed at providing logic reasoning with uncertainty management capabilities. Research in this area encompasses a number of aims, such as the investigation of the source and meaning of uncertainty, the enrichment of logic systems with appropriated formalisms for uncertainty management (e.g., semantics, axioms), and the creation of appropriate models and operators to quantify the propagation of uncertainty in reasoning and inference problems.

Relevant foundational work, with emphasis on analyzing the source and representation of uncertainty in logic systems, can be found in [3]. In this work, the author introduces two different approaches to giving semantics to first-order logics of probability, the first one incorporating probability in the domain (for problems involving statistical information), and the second one assigning probabilities to possible worlds. This work is extended in [4], where the author further discusses the use of a "possible-worlds" framework to represent and reason about uncertainty. Then, quantification of the uncertainty is accomplished by assigning a probability distribution to the possible worlds. In addition, the author discusses the importance of considering time in the inference process, i.e., possible words should describe states at each time point of interest. The work in [5] provides insight on how to process and combine data-driven (e.g., information obtained from observed events) and knowledge-driven (e.g., information provided by domain experts) using different logic systems.

In addition to first-order logic, uncertain representations of logic systems have been extended to other types of logic. For example, the work in [6] introduces a multi-agent epistemic logic able to represent and merge partial beliefs of multiple agents. This logic system is based on possibility theory [7], and enhances epistemic logic with parametric models to obtain lower bounds on the degree of belief of agents. Similarly, an axiomatization of a modal logic using fuzzy sets and DS belief functions for measuring probabilities of modal necessity is presented in [8].

When addressing quantification and propagation of uncertainty in logic reasoning systems, one of the most important approaches is probabilistic logic [9]. Probabilistic logic provides a generalization of logic in which the truth values of sentences are probability values (between 0 and 1). A related approach, possibilistic logic [10], defines mechanisms (based on possibility theory) to associate classical logic formulas with weights. These weights represent lower bounds of necessity degrees. Other approaches that extend logic reasoning to address uncertain scenarios are many-valued and fuzzy logics. Many-valued logics do not restrict the number of truth values of propositions to two. The interpretation of the truth values depends on the actual application. Fuzzy logic can be seen as a type of many-valued logic. Fuzzy logic is based on the theory of fuzzy sets [11]. In fuzzy logic, the imprecision in probabilities is modeled through membership functions defined on the sets of possible probabilities and utilities.

Although useful in some applications, these approaches are sometimes limited by the way they model uncertainty, or simply by the complexity of the problem formulation. Extensions of these approaches could be strengthened by adding more flexibility in assigning probabilities (e.g., through intervals) and a more rigorous method of assigning probability measures (e.g., one that does not require defining priors or membership functions).

Regarding the use of intervals as means of representing uncertainty, it appears in several methods, such as possibility theory [12] and DS theory. The latter, in addition, incorporates a rigorous methodology for assigning probabilistic measures based on available evidence [13]. Given the direct relation that exists between DS theory and probability (DS belief and plausibility measures correspond precisely to probabilistic inner and outer measures [13]), it is possible to simplify DS models to probabilistic models. Considering these advantages, a number of researchers have studied the relation of DS theory and logic. In [14], DS theory is formulated in terms of propositional logic, enabling certain logic reasoning operations in the DS framework. Insight into the relationship between DS theory and probabilistic logic is presented in [14]. A belief-function logic that uses DS models and operations to quantify and estimate uncertainty of logic formulas is introduced in [15]. This logic system allows non-zero belief assignments to the empty set, relies on Dempster's combination rule as the method for quantifying the propagation of uncertainty, and is used in deduction systems where the logic formulas are in Skolemized normal conjunctive form. An application of this system for inference is described in [16]. Further analysis on DS-based logic is presented in [17]. A detailed study on uncertain implication rules is in [18]. This latter work, however, is not focused on ensuring consistency with classical logic, but on modeling causal probabilistic relations.

In spite of existing research to provide logic with uncertainty modeled by DS, efforts to date can be improved by ensuring consistency with classical logic and reducing the number of assumptions needed for the logic systems to work. For example, most of the existing methods are based on Dempster's Combination Rule, which, as it is shown in this manuscript, is not necessarily well suited for logical reasoning. In addition, inference processes could benefit from eliminating the condition that logic formulas need to be expressed in normal conjunctive form or as implication rules, as well as eliminating the need for allowing non-zero belief assignments to the empty set in a DS model.

1.2 Our Contribution: Uncertain Logic

To address these issues, and with emphasis on methods to quantify uncertainty propagation, we introduce uncertain logic. Uncertain logic deals with logic propositions whose truth is uncertain. The level of uncertainty is modeled with DS theory. Uncertain logic allows reasoning and inference using (conventional) first order logic inference rules, but also allows for appending uncertainty to the inference process.

To describe the uncertain logic framework, we start in Section 2 with an overview of DS theory. Basic definitions and notation of uncertain logic are then introduced in Section 3. A set of uncertain logic operators and quantifiers are described in Sections 4 and 5, respectively. Finally, inference in uncertain logic is introduced in Section 6.

2 DS Theory: Basic Definitions

DS Theory is defined for a discrete set of elementary events related to a given problem. This set is called the *Frame of Discernment* (FoD). In general, a FoD is defined as $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$, and has a finite cardinality

$N = |\Theta|$. Elements (or singletons) $\theta_i \in \Theta$ represent the lowest level of discernible information. The power set of Θ is defined as a set containing all the possible subsets of Θ, i.e., $2^\Theta = \{A : A \subseteq \Theta\}$. The cardinality of the power set of Θ is 2^N. Next we introduce some basic definitions of DS Theory, as required for building uncertain logic models. For additional details on DS Theory, we refer the reader to [1, 2].

2.1 Basic Belief Assignment

A Basic Belief Assignment (BBA) or *mass assignment* is a mapping $m_\Theta(\cdot) : 2^\Theta \rightarrow [0, 1]$ such that: $\sum_{A\subseteq\Theta} m_\Theta(A) = 1$ and $m_\Theta(\emptyset) = 0$. The BBA measures the support assigned to proposition $A \subseteq \Theta$. Masses in DS theory can be assigned to any singleton or non-singleton (e.g., $\{\theta_1, \theta_2\}$, $\{\theta_1, \theta_3\}$, $\{\theta_1, \theta_2, \theta_3\}$) proposition. A belief function is called Bayesian if each focal element in Θ is a singleton. The subsets A such that $m(A) > 0$ are referred to as focal elements of the BBA. The set of focal elements is the core $\mathcal{F}_\Theta$. The triple $\{\Theta, \mathcal{F}_\Theta, m_\Theta(\cdot)\}$ is referred to as *Body of Evidence* (BoE).

2.2 Belief and Plausibility

Given a BoE $\{\Theta, \mathcal{F}, m\}$, the *belief* function $\mathrm{Bel} : 2^\Theta \rightarrow [0, 1]$ is defined as: $\mathrm{Bel}_\Theta(A) = \sum_{B\subseteq A} m_\Theta(B)$. $\mathrm{Bel}(A)$ represents the total belief that is committed to A without also being committed to its complement A^C. The *plausibility* function $\mathrm{Pl} : 2^\Theta \rightarrow [0, 1]$ is defined as: $\mathrm{Pl}_\Theta(A) = 1 - \mathrm{Bel}_\Theta(A^C)$. It corresponds to the total belief that does not contradict A. The *uncertainty* of A is: $[\mathrm{Bel}_\Theta(A), \mathrm{Pl}_\Theta(A)]$.

2.3 Combination Rules

Dempster Combination Rule (DCR). For two focal sets $C \subseteq \Theta$ and $D \subseteq \Theta$ such that $B = C \cap D$, and two BBAs $m_j(\cdot)$ and $m_k(\cdot)$, the combined $m_{jk}(B)$ is given by: $m_{jk}(B) = \frac{1}{1-K_{jk}} \sum_{C\cap D=B;B\neq\emptyset} m_j(C)\, m_k(D)$, where $K_{jk} = \sum_{C\cap D=\emptyset} m_j(C)\, m_k(D) \neq 1$ is referred to as the *conflict* between the two BBAs; $K_{jk} = 1$ identifies two totally conflicting BBAs for which DCR-based fusion cannot be carried out.

Conditional Fusion Equation (CFE). A combination rule that is robust when confronted with conflicting evidence is the *Conditional Fusion Equation (CFE)* [19], which is based on the DS theoretic conditional approach [20]. The CFE combines M BBAs as [19]: $m(B) = \sum_{i=1}^{M} \sum_{A_i\in\mathcal{A}_i} \gamma_i(A_i)\, m_i(B|A_i)$, where $\sum_{i=1}^{M} \sum_{A_i\in\mathcal{A}_i} \gamma_i(A_i) = 1$. Here $\mathcal{A}_i = \{A \in \mathcal{F}_i : \mathrm{Bel}_i(A) > 0\}$, $i = 1, \ldots, M$. The conditionals are computed using Fagin-Halperns' Rule of Conditioning [21].

3 From Propositional Logic to Uncertain First-Order Logic

Propositional Logic. Recall that a *proposition* is simply a statement such as "this is an introduction to uncertain logic". We will represent formulas in propositional logic by lower case greek letters (e.g., φ, ψ). In propositional logic, a proposition can be obtained from other propositions using connectives like $\wedge$ (and), $\vee$ (or), $\neg$ (not), and $\Longrightarrow$ (implies). Through (classical) *inference*, propositions can be derived from a given a set of propositions (called *premises*) using (classical) "rules of inference" such as "modus ponens".

Predicate Logic. Predicate logic allows us to look into the structure of propositions. For example, the fact that some entity a is *above* another entity b would be expressed as `Above(a, b)`, where "Above" is a two-place predicate symbol and "`a`" and "`b`" are individual constants. For the remainder of this paper, we will assume finite domains for the interpretation of predicate logic formulas (i.e., individual variables ranges over a finite number of entities).[1]

First-Order Logic. First Order Logic extends predicate logic by the *universal quantifier* ($\forall$) and the *existential quantifier* ($\exists$). Quantified formulas provide a more flexible way of talking about all objects in the domain (i.e., elements in our universe of discourse) or of asserting a property of an individual object.

Uncertain Logic. Uncertain logic deals with propositions $(\varphi_1, \varphi_2, \ldots)$ whose truth is uncertain. The level of uncertainty is modeled with DS theory and is bounded in the range $[0, 1]$. In general, we will consider formulas with k free variables that range over individuals from some finite domain $\Theta_X = \{x_1, \ldots, x_n\}$, with $n \geq 1$, for example

$$\varphi(x), \text{ with uncertainty } [\alpha, \beta], \tag{1}$$

where $\varphi(x)$ is a formula with the only free variable x ranging over elements in Θ_X and $[\alpha, \beta]$ it the corresponding uncertainty interval with $0 \leq \alpha \leq \beta \leq 1$. [2]

To emphasize the fact that uncertain logic models uncertainty of the true value of a proposition, we define the *logical FoD* as follows.

Definition 1 (Logical FoD) *Given a logic proposition $\varphi(x)$ with x ranging over entities in Θ_X, and a true-false FoD $\Theta_{t\text{–}f} = \{\mathbf{1}, \mathbf{0}\}$, the logical FoD $\Theta_{\varphi(x)\times\{\mathbf{1},\mathbf{0}\}}$ is given by:*

$$\Theta_{\varphi(x)\times\{\mathbf{1},\mathbf{0}\}} = \{\varphi(x) \times \mathbf{1}, \varphi(x) \times \mathbf{0}\}. \tag{2}$$

[1]When referring to propositional and predicate logic, we follow the conventions and definitions provided in [22] and [23].

[2]We can define this first-order logic expression more formally as follows: Consider a quantifier-free first-order formula $\varphi(x)$ from a (not necessarily finite) set of formulas Φ in some first-order language L with x being the only free variable in φ. Moreover, let $\Theta_X = \{x_1, \ldots, x_n\}$ be a non-empty set of individuals under observation with respect to formulas in Φ. Throughout this paper, we may represent the logic formula $\varphi(x/x_i)$, the property expressed by φ for the individual $x_i, i = 1, \ldots, n$, with the abbreviated notation $\varphi(x_i)$, i.e., $\varphi(x_i) \equiv \varphi(x/x_i)$. In addition, the DS models that we define for a quantifier-free first-order formula $\varphi(x)$ extend to the sets of formulas $\varphi(x_i), i = 1, \ldots, n$, defined on the corresponding logical FoDs. This extension is used in Section 5, where we define models for existential and universal quantifiers.

When no confusion can arise, we will employ the following notation:

$$\varphi(x) \equiv \varphi(x)\times\mathbf{1}; \varphi(\overline{x}) \equiv \varphi(x)\times\mathbf{0}; \Theta_{\varphi(x)} \equiv \Theta_{\varphi(x)\times\{\mathbf{1},\mathbf{0}\}}.$$

A DS theoretic model that would capture the information in (1) is:

$$\begin{aligned} \varphi(x): \quad m(\varphi(x)) &= \alpha; \\ m(\varphi(\overline{x})) &= 1-\beta; \\ m(\Theta_{\varphi(x)}) &= \beta-\alpha, \end{aligned} \tag{3}$$

defined over the logical FoD $\{\varphi(x), \varphi(\overline{x})\}$. In order to simplify the arguments in the mass assignments, we may use the following alternate notation:

$$\varphi(x): \; m_\varphi(x) = \alpha; m_\varphi(\overline{x}) = 1-\beta; m_\varphi(\Theta_{\varphi(x)}) = \beta-\alpha. \tag{4}$$

Semantics. In classical logic there are two truth values, "true" and "false". An expression that is true for all interpretations is called a tautology ("$\top$"). An expression that is not true for any interpretation is a contradiction ("$\perp$"). Two expressions are semantically equivalent if they take on the same truth value for all interpretations.

In uncertain logic we extend these definitions. The truth value of an expression corresponds to the support that is projected into the true-false FoD, $\Theta_{t\text{-}f} = \{\mathbf{1},\mathbf{0}\}$. A BBA (3) defined by $[\alpha,\beta]=[1,1]$ corresponds to the classical logical truth. A BBA (3) defined by $[\alpha,\beta]=[0,0]$ corresponds to the classical logical falsehood.

The notions of tautology and contradiction in uncertain logic are extended following an approach similar to that in [24]. In particular, given a generic proposition ψ characterized by the uncertainty interval $\sigma=[\alpha,\beta]$, we define a σ-tautology as $\top_\sigma \equiv \psi \vee \neg\psi$, and a σ-contradiction as $\perp_\sigma \equiv \psi \wedge \neg\psi$. It follows that $\top \equiv \top_{\sigma=[1,1]}$, and $\perp \equiv \perp_{\sigma=[0,0]}$.

4 Uncertain Logic Operators

The AND and OR operators are, together with the logical negation, the basic operators in classical logic. This is also the case in uncertain logic, as any other operator can be defined using combinations of these three basic operators. In order to ensure consistency with classical logic, uncertain logic operators should satisfy at least the following: (a) $(\varphi_1(x) \vee \varphi_2(x))$ and $\neg(\neg\varphi_1(x) \wedge \neg\varphi_2(x))$ must have identical DS theoretic models; (b) $(\varphi_1(x) \wedge \varphi_2(x))$ and $\neg(\neg\varphi_1(x) \vee \neg\varphi_2(x))$ have identical DS theoretic models; (c) in the general case, the DS model for AND and OR operations are distinct; (d) in the absence of uncertainty, uncertain logic models converge to those of conventional logic; (e) in a probabilistic scenario (i.e., $\alpha=\beta$), uncertain logic models are also probabilistic; (f) Uncertain logic AND and OR operators must be idempotent, commutative, associative, and distributive.

4.1 Uncertain Logic Negation

Consider a logical FoD $\Theta_{\varphi(x)} = \{\varphi(x), \varphi(\overline{x})\}$ and a BBA $m_\varphi(\cdot)$ defined as:

$$m_\varphi(x) = \alpha;\; m_\varphi(\overline{x}) = 1-\beta;\; m_\varphi(\Theta_{\varphi(x)}) = \beta-\alpha. \tag{5}$$

A complementary BBA for (5) is given by [25]:

$$m^c_\varphi(x) = 1-\beta;\; m^c_\varphi(\overline{x}) = \alpha;\; m^c_\varphi(\Theta_{\varphi(x)}) = \beta-\alpha. \tag{6}$$

Based on the complementary BBA, we can define an uncertain logic negation as follows.

Definition 2 (Logical Not in Uncertain Logic) *Given an uncertain proposition $\varphi(x)$ as defined in (1), and its corresponding DS model defined by (4), the logical negation of $\varphi(x)$ is given by:*

$$\neg\varphi(x), \text{ with uncertainty } [1-\beta, 1-\alpha]. \tag{7}$$

We utilize the complementary BBA corresponding to (4) as the DS theoretic model for $\neg\varphi(x)$, i.e.,

$$\begin{aligned} \neg\varphi(x): \quad m^c_\varphi(x) &= 1-\beta; \\ m^c_\varphi(\overline{x}) &= \alpha; \\ m^c_\varphi(\Theta_{\varphi(x)}) &= \beta-\alpha. \end{aligned} \tag{8}$$

Definition 2 satisfies an important property: Given a proposition $\varphi(x)$, the BBA corresponding to its double-negation is the same model as the one associated with $\varphi(x)$. In other words, Definition 2 satisfies $\neg\neg\varphi(x) = \varphi(x)$, which is a basic property in (classical) logic.

4.2 Uncertain Logic AND/OR

Definition 3 (Logical And & Or in Uncertain Logic) *Suppose that we have M logic propositions, each providing a statement of the following type regarding the truth of x with respect to the proposition $\varphi_i(\cdot)$:*

$$\varphi_i(x), \text{ with uncertainty } [\alpha_i, \beta_i],\; i=1,\ldots,M. \tag{9}$$

The corresponding DS theoretic models are $\varphi_i(x): m_{\varphi_i}(x) = \alpha_i$; $m_{\varphi_i}(\overline{x}) = 1-\beta_i$; $m_{\varphi_i}(\Theta_{\varphi_i(x)}) = \beta_i - \alpha_i$, for $i=1,2,\ldots,M$. We propose to utilize the following DS theoretic models for the logical AND and OR of the statements in (9)*:*

$$\begin{aligned} &\bigwedge_{i=1}^{M} \varphi_i(x): \quad m(\cdot) = \bigcap_{i=1}^{M} m_{\varphi_i}(\cdot); \\ \text{and} \quad &\bigvee_{i=1}^{M} \varphi_i(x): \quad m(\cdot) = \left(\bigcap_{i=1}^{M} m^c_{\varphi_i}(\cdot)\right)^c, \end{aligned} \tag{10}$$

where $\bigcap$ denotes an appropriate fusion operator.[3]

[3]A similar model can be obtained for the case of AND/OR operations of a set of expressions $\{\varphi(x_i)\}$ with uncertainty $[\alpha_i,\beta_i]$, $x_i \in \{x_1, x_2, \ldots, x_n\}$. In this case, $\bigwedge_{i=1}^n \varphi(x_i): \; m(\cdot) = \bigcap_{i=1}^n m_\varphi(\cdot)$, and $\bigvee_{i=1}^n \varphi(x_i): \; m(\cdot) = \left(\bigcap_{i=1}^n m^c_\varphi(\cdot)\right)^c$. This case represents AND/OR models applied to the truthfulness of elements $\{x_i\}$ satisfying a property φ, whereas (10) analyzes the case of x satisfying multiple properties $\{\varphi_i\}$.

Table 1: DCR-Based Logical AND and OR. Note that the DS models for AND and OR are identical, which suggests that DCR is not an appropriate fusion operator for consistent logic operations. Note that, in both cases, the masses should be normalized by $1-K$, with $K=1-\sum_{A\in\mathcal{F}} m(A)=\alpha_1(1-\beta_2)+(1-\beta_1)\alpha_2$.

Focal Set	$\varphi_1(x)\wedge\varphi_2(x)$	$\varphi_1(x)\vee\varphi_2(x)$
x	$\alpha_1\beta_2+(\beta_1-\alpha_1)\alpha_2$	$\alpha_1\beta_2+(\beta_1-\alpha_1)\alpha_2$
$\overline{x}$	$(1-\beta_1)(1-\alpha_2)+(\beta_1-\alpha_1)(1-\beta_2)$	$(1-\beta_1)(1-\alpha_2)+(\beta_1-\alpha_1)(1-\beta_2)$
$\Theta_{(\varphi_1\cdot\varphi_2)(x)}$	$(\beta_1-\alpha_1)(\beta_2-\alpha_2)$	$(\beta_1-\alpha_1)(\beta_2-\alpha_2)$

4.3 DCR-Based Uncertain Logic

When the fusion operator $\bigcap$ in (10) is DCR, the AND operation in this model is equivalent to the conjunctive rule of combination in [17]. In this subsection we go further and explore the viability of using DCR as the fusion operator in uncertain logic.

Consider the two-source/two-propositions (*i.e.*, $M=2$) case. Table 1 contains the DCR-based logical AND and OR operations for this case. Notice that the mass assignments for the AND operation (*i.e.*, $\varphi_1(x)\wedge\varphi_2(x)$) are exactly the same as the ones obtained for the OR operation (*i.e.*, $\varphi_1(x)\vee\varphi_1(x)$). Having identical models for both AND and OR operators suggests that, although DCR may work as a fusion operator for certain operations, it does not render models that satisfy important properties for all the logical operations defined in this paper. More particularly, DCR-based uncertain logic does not satisfy the "uniqueness of the model" property. As an alternative, we propose using a more appropriate fusion strategy, such as the CFE, which is analyzed next.

4.4 CFE-Based Uncertain Logic

Recall (from Section 2) that CFE-based fusion requires the definition of coefficients $\gamma_i(\cdot)$. For uncertain logic, we introduce the Logic Consistent (LC) strategy, which ensures consistency with logical operations.

Definition 4 (Logic Consistent (LC) Strategy) *For the case $M=2$ in (10), let us define $\underline{\alpha}=\min(\alpha_1,\alpha_2)$; $\underline{\beta}=\min(\beta_1,\beta_2)$; $\overline{\alpha}=\max(\alpha_1,\alpha_2)$; $\overline{\beta}=\max(\beta_1,\beta_2)$; $\delta_1=\beta_1-\alpha_1$; $\delta_2=\beta_2-\alpha_2$; $\underline{\delta}=\underline{\beta}-\underline{\alpha}$; and $\overline{\delta}=\overline{\beta}-\overline{\alpha}$. Then select the CFE parameters as follows:*

$$\gamma_1(x)=\gamma_2(x)\equiv\gamma(x);\quad \gamma_1(\overline{x})=\gamma_2(\overline{x})\equiv\gamma(\overline{x});$$
$$\gamma_1(\Theta)=\gamma_2(\Theta)\equiv\gamma(\Theta),$$

where the CFE parameters $\gamma(x)$, $\gamma(\overline{x})$, and $\gamma(\Theta)$ are selected in the following manner.

a. Logical AND:

- *If $\delta_1+\delta_2\neq 0$:*

$$\gamma(x)=\frac{\underline{\alpha}(\beta_1+\beta_2)-\underline{\beta}(\alpha_1+\alpha_2)}{2(\delta_1+\delta_2)};$$
$$\gamma(\overline{x})=\frac{1}{2}-\frac{\underline{\beta}(2-\alpha_1-\alpha_2)-\underline{\alpha}(2-\beta_1-\beta_2)}{2(\delta_1+\delta_2)};$$
$$\gamma(\Theta)=\frac{\underline{\delta}}{\delta_1+\delta_2}.$$

- *If $\delta_1+\delta_2=0$, i.e., $\alpha_1=\beta_1$ and $\alpha_2=\beta_2$:*

$$\gamma(x)=\frac{\underline{\alpha}-\gamma(\Theta)\,(\alpha_1+\alpha_2)}{2};$$
$$\gamma(\overline{x})=\frac{(1-\underline{\alpha})-\delta(\Theta)\,(2-\alpha_1-\alpha_2)}{2};$$
$$\gamma(\Theta)=\textit{arbitrary}.$$

b. Logical OR:

- *If $\delta_1+\delta_2\neq 0$:*

$$\gamma(x)=\frac{1}{2}-\frac{\overline{\beta}(2-\alpha_1-\alpha_2)-\overline{\alpha}(2-\beta_1-\beta_2)}{2(\delta_1+\delta_2)};$$
$$\gamma(\overline{x})=\frac{\overline{\alpha}(\beta_1+\beta_2)-\overline{\beta}(\alpha_1+\alpha_2)}{2(\delta_1+\delta_2)};$$
$$\gamma(\Theta)=\frac{\overline{\delta}}{\delta_1+\delta_2}.$$

- *If $\delta_1+\delta_2=0$, i.e., $\alpha_1=\beta_1$ and $\alpha_2=\beta_2$:*

$$\gamma(x)=\frac{\overline{\alpha}-\gamma(\Theta)\,(\alpha_1+\alpha_2)}{2};$$
$$\gamma(\overline{x})=\frac{(1-\overline{\alpha})-\delta(\Theta)\,(2-\alpha_1-\alpha_2)}{2};$$
$$\gamma(\Theta)=\textit{arbitrary}.$$

When used for the AND operation, the LC strategy renders the following BBA (see Appendix A for the derivation of this BBA):

$$\varphi_1(x)\wedge\varphi_2(x):\quad m(x)=\underline{\alpha};\quad m(\overline{x})=1-\underline{\beta};\ \text{and}\quad m(\Theta_{(\varphi_1\wedge\varphi_2)(x)})=\underline{\beta}-\underline{\alpha}. \tag{11}$$

When used for the OR operation, the LC strategy renders the following BBA:

$$\varphi_1(x)\vee\varphi_2(x):\quad m(x)=\overline{\alpha};\quad m(\overline{x})=1-\overline{\beta};\ \text{and}\quad m(\Theta_{(\varphi_1\vee\varphi_2)(x)})=\overline{\beta}-\overline{\alpha}. \tag{12}$$

In general, the CFE-based models for the logical AND and OR are not identical (the exception would be a particular combination of uncertainty parameters $[\alpha,\beta]$ rendering identical models), as is the case when DCR is used. Therefore, CFE-based fusion is better suited for uncertain logic than DCR. Indeed, referring to the conditions at the beginning of Section 4, the CFE-based operations are *consistent*

Table 2: CFE-Based AND/OR Operations: Uncertainty parameters are defined so that they represent complete certainty on the truth (or falseness) of each proposition.

Parameters		$m_{\varphi_1 \wedge \varphi_2}(\cdot)$			$m_{\varphi_1 \vee \varphi_2}(\cdot)$		
$[\alpha_1, \beta_1]$	$[\alpha_2, \beta_2]$	x	$\overline{x}$	Θ	x	$\overline{x}$	Θ
$[0, 0]$	$[0, 0]$	0	1	0	0	1	0
$[0, 0]$	$[1, 1]$	0	1	0	1	0	0
$[1, 1]$	$[1, 1]$	1	0	0	1	0	0

Table 3: CFE-Based Logical AND/OR Operations: Probabilistic Scenario ($[\alpha_i, \beta_i] = [\alpha_i, \alpha_i],\ i \in \{1, 2\}$).

Logical AND	Logical OR
$m(x) = \underline{\alpha}$	$m(x) = \overline{\alpha}$
$m(\overline{x}) = 1 - \underline{\alpha}$	$m(\overline{x}) = 1 - \overline{\alpha}$
$m(\Theta_{(\varphi_1 \wedge \varphi_2)(x)}) = 0$	$m(\Theta_{(\varphi_1 \vee \varphi_2)(x)}) = 0$

with classical logic. Referring to the same conditions, (a) and (b) can be verified by checking Definition 3; (c) is verified by (11) and (12) above; (d) is proved in Table 2; (e) is shown in Table 3; (f) is proved in Appendix B.

4.5 Other Uncertain Logic Operators

Based on the uncertain logic definitions and operators described above, it is possible to extend them and create new operators. As an example, consider implication rules.

Definition 5 (Logical Implication in Uncertain Logic) *Given two logic statements $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$, an implication rule in propositional logic has the property:*

$$\begin{aligned}\varphi_1(x_i) \implies \varphi_2(y_j) &= \neg\varphi_1(x_i) \vee \varphi_2(y_j) \\ &= \neg\left(\varphi_1(x_i) \wedge \neg\varphi_2(y_j)\right),\end{aligned}$$

where $x_i \in \Theta_X$ and $y_j \in \Theta_Y$. Consider the case where the antecedent $\varphi_1(x_i)$ and/or the consequent $\varphi_2(y_j)$ are/is uncertain. Furthermore, suppose that said uncertainty is represented via the DS theoretic models $m_X(\cdot)$ and $m_Y(\cdot)$ over the logical FoDs $\{\varphi(x_i), \varphi(\overline{x_i})\}$ and $\{\varphi(y_j), \varphi(\overline{y_j})\}$, respectively. Then, the implication rule $\varphi_1(x_i) \implies \varphi_2(y_j)$ is taken to have the following DS theoretic model:

$$\begin{aligned}m_{\varphi_X \to \varphi_Y}(\cdot) &= (m_X^c \vee m_Y)(\cdot) \\ &= (m_X \wedge m_Y^c)^c(\cdot), \qquad (13)\end{aligned}$$

over the FoD $\{\varphi(x_i), \varphi(\overline{x_i})\} \times \{\varphi(y_j), \varphi(\overline{y_j})\}$.

5 Uncertain Logic Quantifiers

We define existential and universal quantifiers in uncertain logic as follows.

Definition 6 (Existential Quantifier in Uncertain Logic) *Consider the statement:*

$$\exists x \ \ \varphi(x), \ \textit{with uncertainty}\ [\alpha, \beta], \qquad (14)$$

where $x \in \Theta_X = \{x_1, x_2, \ldots, x_N\}$. Let us define an extended logical FoD $\Theta_{X'} = \{\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_N)\} \times \{\mathbf{1}, \mathbf{0}\}$. Then, we define the DS theoretic model for (14) as:

$$\bigvee_{i=1}^{N} \varphi(x_i), \qquad (15)$$

over the FoD $\Theta_{X'}$, subject to the constraint:

$$\begin{aligned}m(\mathbf{1}) &= \sum\nolimits_{i=1}^{N} m_\varphi(x_i) = \alpha; \\ m(\mathbf{0}) &= \sum\nolimits_{i=1}^{N} m_\varphi(\overline{x_i}) = 1 - \beta; \\ m(\Theta_{X'}) &= \beta - \alpha. \qquad (16)\end{aligned}$$

This model is an alternative to Skolemization [23]. This model, however, does not rule out the use of Skolemization, as there might be scenarios where the latter technique is a better alternative. Note that if the uncertainty of at least one of the propositions $\varphi(x_i)$ in (15) is $[\alpha, \beta]$, and the uncertainty of every other proposition is $[0, 0]$ (or, in general, $[\alpha_j, \beta_j]$, with $\alpha_j \leq \alpha$, $\beta_j \leq \beta$, and $i \neq j$), then the DS model corresponding to (15) is equivalent to the DS model corresponding to (14) when the OR operations are computed as indicated by Definitions 3 and 4. Also, although an infinite number of solutions satisfy (16), a useful solution (e.g., for existential instantiation on inference problems) is given by $m_\varphi(x_i) = \alpha$; $m_\varphi(\overline{x_i}) = 1 - \beta$; and $m_\varphi(\{x_i, \overline{x_i}\}) = \beta - \alpha, i = 1, 2, \ldots, N$. This solution can be proven by successively applying the idempotency property to the OR operator.

Definition 7 (Universal Quantifier in Uncertain Logic) *Consider the statement:*

$$\forall x \ \ \varphi(x), \ \textit{with uncertainty}\ [\alpha, \beta], \qquad (17)$$

where $x \in \Theta_X = \{x_1, x_2, \ldots, x_N\}$. Then, we define the DS theoretic model for (17) as:

$$\bigwedge_{i=1}^{N} \varphi(x_i), \qquad (18)$$

over the FoD $\Theta_{X'} = \{\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_N)\} \times \{\mathbf{1}, \mathbf{0}\}$, subject to the constraint:

$$\begin{aligned}m(\mathbf{1}) &= \sum\nolimits_{i=1}^{N} m_\varphi(x_i) = \alpha; \\ m(\mathbf{0}) &= \sum\nolimits_{i=1}^{N} m_\varphi(\overline{x_i}) = 1 - \beta; \\ m(\Theta_{X'}) &= \beta - \alpha. \qquad (19)\end{aligned}$$

Note that if the uncertainty of every proposition $\varphi(x_i)$ in (18) is $[\alpha, \beta]$, then the DS model corresponding to (18) is equivalent to the DS model corresponding to (17) when the AND operations are computed as indicated by Definitions 3 and 4. Also, although an infinite number of solutions satisfy (19), a useful solution (e.g., for universal instantiation on inference) is given by $m_\varphi(x_i) = \alpha$; $m_\varphi(\overline{x_i}) = 1 - \beta$; and $m_\varphi(\{x_i, \overline{x_i}\}) = \beta - \alpha, i = 1, 2, \ldots, N$. This solution can be proven by applying idempotency to the AND operator.

6 Inference in Uncertain Logic

Inference in uncertain logic shares the fundamental principles of classical logic, and adds the possibility of attaching, tracking, and propagating uncertainties that may arise on premises and/or rules. Due to the extensive number of methods for logic inference, the scope of this section is limited to the introduction of some of the most fundamental inference rules, along with some basic examples that illustrate uncertain logic inference. For an extended definition of these rules and their application for inference in the context of classical logic, we refer the reader to [22].

Modus Ponens (MP). This rule states that, whenever the logic sentences $\varphi \implies \psi$ and φ have been established, then it is acceptable to infer the sentence ψ as well. MP extends to uncertain logic as follows. Consider:

$$\begin{aligned} \varphi_1(x)&, \text{ with uncertainty } [\alpha_1, \beta_1]; \\ \varphi_2(y)&, \text{ with uncertainty } [\alpha_2, \beta_2]; \text{ and} \\ \varphi_1(x) \implies \varphi_2(y)&, \text{ with uncertainty } [\alpha_R, \beta_R]. \end{aligned} \quad (20)$$

Then, given the uncertain premises $\varphi_1(x) \implies \varphi_2(y)$ and φ_1, MP allows us to infer the uncertain expression $\varphi_2(y)$. Note that, if the uncertainty parameters $[\alpha_2, \beta_2]$ are unknown, their value should be obtained by applying the methodology introduced in Section 4 above. It can be shown that uncertain MP (as well as the inference rules introduced this section) lead to $\top_\sigma$, with $\sigma = [\max(\alpha_R, 1-\beta_R), \max(\alpha_R, 1-\beta_R)]$.

To better understand MP in uncertain logic, consider an example where $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$. By using the model in Definition 5, we can obtain $\alpha_R = \beta_R = 1$. Furthermore, given the $\varphi_1(x) \implies \varphi_2(y)$ and $\varphi_1(x)$, then we can infer $\varphi_2(y)$ with uncertainty $[\alpha_2 = \beta_2] = [1, 1]$. This case represents a scenario with no uncertainty.

Now consider a scenario where there is uncertainty in the rule, in such a way that $[\alpha_R, \beta_R] = [0.5, 1.0]$, and assume that we have a model for the uncertainty of $\varphi_1(x)$ such that $\alpha_1 = \beta_1 = 1$. Then, MP allows us to infer $\varphi_2(y)$, with the uncertainty $[\alpha_2, \beta_2]$ obtained from the equations $\alpha_R = \max(1-\beta_1, \alpha_2)$ and $\beta_R = \max(1-\alpha_1, \beta_2)$. Solving these equations we obtain $\alpha_2 = 0.5$ and $\beta_2 = 1$.

Modus Tolens (MT). This rule states that, if we know that $\varphi \implies \psi$, then we can infer $\neg\varphi$ if we believe that ψ is false. MT extends to uncertain logic as follows. Assume that the uncertainty on each of the expressions involved in MP are defined by (20). Then, given the uncertain premises $\varphi_1(x) \implies \varphi_2(y)$ and $\neg\varphi_2$, MT allows us to infer the uncertain expression $\neg\varphi_1(y)$. As with MP above, if the uncertainty parameters $[\alpha_2, \beta_2]$ are unknown, their value should be obtained by applying the methodology introduced in Section 4.

Other rules of inference. Uncertain logic can be extended by incorporating new rules of inference that already exist in conventional logic inference. Some examples of new rules of inference are: AND elimination (AE), AND introduction (AI), universal instantiation (UI), and existential instantiation (EI). The definition of these rules of inference is straightforward based on their definition for conventional logic, and is not included in this manuscript.

Example. Consider the following problem, originally introduced in [22]. We know that horses are faster than dogs and that there is a greyhound that is faster than every rabbit. We know that Harry is a horse and that Ralph is a rabbit. We also know that greyhounds are dogs and that our speed relationship is transitive. Then:

$$\forall x\, \forall y\ \text{Horse}(x) \wedge \text{Dog}(y) \Rightarrow \text{Faster}(x, y) \quad (21a)$$

$$\exists y\ \text{Greyhound}(y) \wedge (\forall z\ \text{Rabbit}(z) \Rightarrow \text{Faster}(y, z)) \quad (21b)$$

$$\forall y\ \text{Greyhound}(y) \Rightarrow \text{Dog}(y) \quad (21c)$$

$$\forall x\, \forall y\, \forall z\ \text{Faster}(x, y) \wedge \text{Faster}(y, z) \Rightarrow \text{Faster}(x, z) \quad (21d)$$

$$\text{Horse}(\text{Harry}) \quad (21e)$$

$$\text{Rabbit}(\text{Ralph}). \quad (21f)$$

Using these logic statements, it can be inferred that Harry is faster than Ralph (i.e., Faster(Harry, Ralph)) [22].

Now, let us introduce uncertain logic operations by assuming that the logic premise (21a) is uncertain, with uncertainty $[\alpha_1, \beta_1]$, and that there is no uncertainty in premises (21b)-(21f). This represents some uncertainty in the sentence "horses are faster than dogs", which may occur if we consider cases such as sick or old horses compared to healthy dogs. The steps that are used for inferring Faster(Harry, Ralph), as well as the uncertainty in each of the steps of this process are in Table 4. It is easy to verify that, if $\alpha_1 = \beta_1 = 1$. The initial steps in the inference process are simply the reproduction of (21a)-(21f) as premises 1 to 6. Steps 7 to 13 can be obtained from applying EI, AI, UI, and MP rules to premises 2 to 6. In our initial example (only the first premise is uncertain), the uncertainty in premises 2 to 6 is $[\alpha_i, \beta_i] = [1, 1], i = 2, 3, \ldots, 6$. Uncertain logic operations become relevant in steps 14 to 19. For example, the uncertainty in premise 16 is obtained from solving the system of equations shown in the corresponding row in Table 4. This system of equations is derived from Definition 5. As a consequence, any change in the uncertainty $[\alpha_1, \beta_1]$ directly affects $[\alpha_{16}, \beta_{16}]$. Figure 1 illustrates the result in a probabilistic scenario. Note that, for us to be able to conclude "Faster(Harry, Ralph)" given the initial uncertainty, α_4 must be larger than α_1. Similar results can be further verified by modifying uncertainties on the premises, whose values can be computed as indicated in Table 4.

7 Conclusions

We have introduced *Uncertain Logic*, a DS theoretic approach for first order logic operations. Uncertain logic provides support for handling variables and quantifiers, in addition to fundamental logic operations (i.e., $\neg, \wedge, \vee$). The framework introduced in this paper allows systematic generation of mass assignments based on uncertain

Table 4: Steps followed for the inference of the sentence Faster(Harry, Ralph) based on the premises defined in (21). The uncertainty is obtained from applying uncertain logic definitions and rules to the example described in Section 6.

	Logic Formula	Premises & Rule	Uncertainty
1	$\forall x\, \forall y$ Horse$(x) \wedge$ Dog$(y) \Rightarrow$ Faster(x, y)	Δ	$[\alpha_1, \beta_1]$
2	$\exists y$ Greyhound$(y) \wedge (\forall z$ Rabbit$(z) \Rightarrow$ Faster$(y, z))$	Δ	$[\alpha_2, \beta_2] = [1, 1]$
3	$\forall y$ Greyhound$(y) \Rightarrow$ Dog(y)	Δ	$[\alpha_3, \beta_3] = [1, 1]$
4	$\forall x\, \forall y\, \forall z$ Faster$(x, y) \wedge$ Faster$(y, z) \Rightarrow$ Faster(x, z)	Δ	$[\alpha_4, \beta_4]$
5	Horse(Harry)	Δ	$[\alpha_5, \beta_5]$
6	Rabbit(Ralph)	Δ	$[\alpha_6, \beta_6] = [1, 1]$
7	Greyhound(Greg) $\wedge (\forall z$ Rabbit$(z) \implies$ Faster(Greg, z))	2, EI	$[\alpha_7, \beta_7] = [1, 1]$
8	Greyhound(Greg)	7, AE	$[\alpha_8, \beta_8] = [1, 1]$
9	$\forall z$ Rabbit$(z) \implies$ Faster(Greg, z)	7, AE	$[\alpha_9, \beta_9] = [1, 1]$
10	Rabbit(Ralph) $\implies$ Faster(Greg, Ralph)	9, UI	$[\alpha_{10}, \beta_{10}] = [1, 1]$
11	Faster(Greg, Ralph)	10, 6, MP	$[\alpha_{11}, \beta_{11}] = [1, 1]$
12	Greyhound(Greg) $\implies$ Dog(Greg)	3, UI	$[\alpha_{12}, \beta_{12}] = [1, 1]$
13	Dog(Greg)	12, 8, MP	$[\alpha_{13}, \beta_{13}] = [1, 1]$
14	Horse(Harry) $\wedge$ Dog(Greg) $\implies$ Faster(Harry, Greg)	1, UI	$[\alpha_{14}, \beta_{14}] = [\alpha_1, \beta_1]$
15	Horse(Harry) $\wedge$ Dog(Greg)	5, 13, AI	$[\alpha_{15}, \beta_{15}] = [\alpha_5, \beta_5]$
16	Faster(Harry, Greg)	14, 15, MP	$[\alpha_{16}, \beta_{16}]$ obtained from solving $\begin{cases}\alpha_{14} = \max(1-\beta_{15}, \alpha_{16}) \\ \beta_{14} = \max(1-\alpha_{15}, \beta_{16})\end{cases}$
17	Faster(Harry, Greg) $\wedge$ Faster(Greg, Ralph) $\implies$ Faster(Harry, Ralph)	4, UI	$[\alpha_{17}, \beta_{17}] = [\alpha_4, \beta_4]$
18	Faster(Harry, Greg) $\wedge$ Faster(Greg, Ralph)	16, 11, AI	$[\alpha_{18}, \beta_{18}] = [\alpha_{16}, \beta_{16}]$
19	Faster(Harry, Ralph)	17, 18, MP	$[\alpha_{19}, \beta_{19}]$ obtained from solving $\begin{cases}\alpha_{17} = \max(1-\beta_{18}, \alpha_{19}) \\ \beta_{17} = \max(1-\alpha_{18}, \beta_{19})\end{cases}$

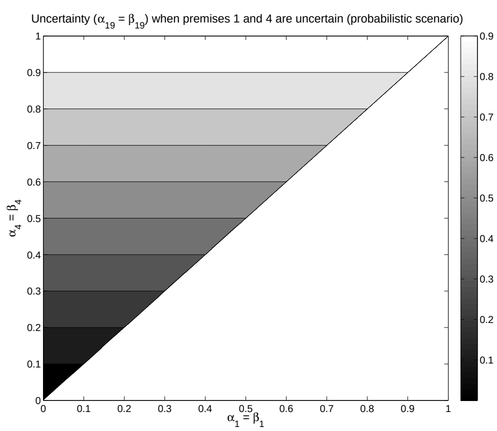

Figure 1: Uncertainty in Premise 19 of Table 4.

first order logic formulas. Furthermore, by using appropriate fusion operators, higher-level applications are possible within this framework, such as inference and resolution based on uncertain data models.

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Appendix A. BBA for LC CFE-based AND

Based on the definition of the CFE fusion operator:

$$m(x) = \gamma_1(x) + \gamma_1(\Theta) m_1(x) + \gamma_2(x) + \gamma_2(\Theta) m_2(x). \quad (22)$$

Substituting the CFE coefficients for the AND operation, as indicated by Definition 4, in (22):

$$m(x) = 2\gamma(x) + 2\gamma(\Theta)(\alpha_1 + \alpha_2).$$

- When $\delta_1 + \delta_2 \neq 0$:

$$\begin{aligned} m(x) &= \frac{\underline{\alpha}(\beta_1+\beta_2) - \underline{\beta}(\alpha_1+\alpha_2)}{\delta_1+\delta_2} + \frac{\underline{\delta}(\alpha_1+\alpha_2)}{\delta_1+\delta_2} \\ &= \tfrac{1}{\delta_1+\delta_2}(\underline{\alpha}\beta_1 + \underline{\alpha}\beta_2 - \alpha_1\underline{\beta} - \alpha_2\underline{\beta} + \underline{\delta}(\alpha_1+\alpha_2)). \end{aligned}$$

  Since $\underline{\delta} = \underline{\beta} - \underline{\alpha}$:

$$\begin{aligned} m(x) &= \tfrac{1}{\delta_1+\delta_2}(\underline{\alpha}\beta_1 + \underline{\alpha}\beta_2 - \alpha_1\underline{\beta} - \alpha_2\underline{\beta} \\ &\quad + \alpha_1\underline{\beta} + \alpha_2\underline{\beta} - \alpha_1\underline{\alpha} - \alpha_2\underline{\alpha}) \\ &= \tfrac{1}{\delta_1+\delta_2}(\underline{\alpha}\beta_1 + \underline{\alpha}\beta_2 - \alpha_1\underline{\alpha} - \alpha_2\underline{\alpha}) \\ &= \tfrac{1}{\delta_1+\delta_2}(\underline{\alpha}(\beta_2 - \alpha_2 + \beta_1 - \alpha_1)). \quad (23) \end{aligned}$$

  Substituting $\delta_1 = \beta_1 - \alpha_1$ and $\delta_2 = \beta_2 - \alpha_2$ in (23): $m(x) = \underline{\alpha}$.

- When $\delta_1 + \delta_2 = 0$, and making $\gamma(\Theta) = 0$:

$$m(x) = 2\gamma(x) = \underline{\alpha}.$$

The mass $m(\overline{x})$ is given by:

$$m(\overline{x}) = \gamma_1(\overline{x}) + \gamma_1(\Theta) m_1(\overline{x}) + \gamma_2(\overline{x}) + \gamma_2(\Theta) m_2(\overline{x}). \quad (24)$$

Substituting the CFE coefficients as indicated by Definition 4 in (24):

$$m(\overline{x}) = 2\gamma(\overline{x}) + 2\gamma(\Theta)(2 - \beta_1 - \beta_2).$$

- When $\delta_1 + \delta_2 \neq 0$:

$$\begin{aligned} m(\overline{x}) &= \frac{\delta_1 + \delta_2 - \underline{\beta}(2-\alpha_1-\alpha_2) + \underline{\alpha}(2-\beta_1-\beta_2)}{\delta_1+\delta_2} \\ &\quad + \frac{\underline{\delta}(2-\beta_1-\beta_2)}{\delta_1+\delta_2} \\ &= \tfrac{1}{\delta_1+\delta_2}(\delta_1 + \delta_2 - \underline{\beta}(2-\alpha_1-\alpha_2) \\ &\quad + \underline{\alpha}(2-\beta_1-\beta_2) + \underline{\delta}(2-\beta_1-\beta_2)). \end{aligned}$$

Since $\underline{\delta} = \underline{\beta} - \underline{\alpha}$:

$$m(\overline{x}) = \frac{1}{\delta_1+\delta_2}(\delta_1 + \delta_2 - \underline{\beta}(2 - \alpha_1 - \alpha_2) + \underline{\alpha}(2 - \beta_1 - \beta_2) + (\underline{\beta} - \underline{\alpha})(2 - \beta_1 - \beta_2))$$
$$= \frac{1}{\delta_1+\delta_2}(\delta_1 + \delta_2 - \underline{\beta}(2 - \alpha_1 - \alpha_2 - 2 + \beta_1 + \beta_2))$$
$$= \frac{1}{\delta_1+\delta_2}(\delta_1 + \delta_2 - \underline{\beta}(\beta_1 - \alpha_1 + \beta_2 - \alpha_2)).$$

Substituting $\delta_1 = \beta_1 - \alpha_1$ and $\delta_2 = \beta_2 - \alpha_2$ in (24):

$$m(x) = 1 - \underline{\beta}.$$

- When $\delta_1 + \delta_2 = 0$, and making $\gamma(\Theta) = 0$:

$$m(\overline{x}) = 2\gamma(\overline{x}) = 1 - \underline{\alpha} = 1 - \underline{\beta}.$$

Finally, $m(\Theta) = 1 - m(x) - m(\overline{x}) = \underline{\beta} - \underline{\alpha}$.

Appendix B. Properties of the LC CFE-based Uncertain Logic operations

Consider logic expressions of the form $\varphi(x_i)$, with $1 \le i \le N$. Then, the following properties are satisfied:

1. *Idempotency*: This property is defined by: $\varphi_i(x) \wedge \varphi_i(x) = \varphi_i(x) \vee \varphi_i(x) = \varphi_i(x)$. In this case:

$$m_\wedge(x) = \underline{\alpha} = \min(\alpha_i, \alpha_i) = \alpha_i = \max(\alpha_i, \alpha_i) = \overline{\alpha} = m_\vee(x);$$
$$m_\wedge(\overline{x}) = 1 - \underline{\beta} = 1 - \min(\beta_i, \beta_i) = 1 - \beta_i = 1 - \max(\beta_i, \beta_i) = 1 - \overline{\beta} = m_\vee(\overline{x});$$
$$m_\wedge(\Theta) = \underline{\beta} - \underline{\alpha} = \beta_i - \alpha_i = \overline{\beta} - \overline{\alpha} = m_\vee(\Theta).$$

2. *Commutativity*: This property refers to satisfying: $\varphi_1(x) \wedge \varphi_2(x) = \varphi_2(x) \wedge \varphi_1(x)$, and $\varphi_1(x) \vee \varphi_2(x) = \varphi_2(x) \vee \varphi_1(x)$. Let us call $m_{\varphi_i \wedge \varphi_j}(\cdot)$ the BBA resulting from $\varphi_i(x) \wedge \varphi_j(x)$, $i = \{1, 2\}$. Then, for the AND operation:

$$m_{\varphi_1\wedge\varphi_2}(x) = \min(\alpha_1, \alpha_2) = \min(\alpha_2, \alpha_1) = m_{\varphi_2\wedge\varphi_1}(x)$$
$$m_{\varphi_1\wedge\varphi_2}(\overline{x}) = 1 - \min(\beta_1, \beta_2) = 1 - \min(\beta_2, \beta_1) = m_{\varphi_2\wedge\varphi_1}(\overline{x})$$
$$m_{\varphi_1\wedge\varphi_2}(\Theta) = \min(\beta_1, \beta_2) - \min(\alpha_1, \alpha_2) = \min(\beta_2, \beta_1) - \min(\alpha_2, \alpha_1) = m_{\varphi_2\wedge\varphi_1}(\Theta).$$

   A proof for commutativity for the logical OR operation is obtained by following a similar procedure.

3. *Associativity*: The associative property is defined by: $\varphi_1(x) \wedge [\, \varphi_2(x) \wedge \varphi_3(x)\,] = [\varphi_1(x) \wedge \varphi_2(x)\,] \wedge \varphi_3(x)$, and $\varphi_1(x) \vee [\, \varphi_2(x) \vee \varphi_3(x)\,] = [\, \varphi_1(x) \vee \varphi_2(x)\,] \vee \varphi_3(x)$. Let us call $\varphi_4(\cdot)$ the model generated by $\varphi_2(x) \wedge \varphi_3(x)$, and $\varphi_5(\cdot)$ the model generated by $\varphi_1(x) \wedge \varphi_2(x)$. Also, let us call $m_{\varphi_i\wedge\varphi_j}(\cdot)$ the BBA resulting from $\varphi_i(x) \wedge \varphi_j(x)$, $i = \{1, \ldots, 5\}$. Our goal (for the AND operation) is to show that the model for $\varphi_1(\cdot) \wedge \varphi_4(\cdot)$ is equivalent to the model for $\varphi_5(\cdot) \wedge \varphi_3(\cdot)$:

$$m_{\varphi_1\wedge\varphi_4}(x) = \min(\alpha_1, \min(\alpha_2, \alpha_3)) = \min(\min(\alpha_1, \alpha_2), \alpha_3) = m_{\varphi_5\wedge\varphi_3}(x)$$
$$m_{\varphi_1\wedge\varphi_4}(\overline{x}) = 1 - \min(\beta_1, \min(\beta_2, \beta_3)) = 1 - \min(\min(\beta_1, \beta_2), \beta_3) = m_{\varphi_5\wedge\varphi_2}(\overline{x})$$
$$m_{\varphi_1\wedge\varphi_4}(\Theta) = \min(\beta_1, \min(\beta_2, \beta_3)) - \min(\alpha_1, \min(\alpha_2, \alpha_3)) = \min(\min(\beta_1, \beta_2), \beta_3) - \min(\min(\alpha_1, \alpha_2), \alpha_3) = m_{\varphi_5\wedge\varphi_3}(\Theta).$$

   A proof for associativity for the logical OR operation is obtained by following a similar procedure.

4. *Distributivity*: Distributive operations satisfy: $\varphi_1(x_i) \wedge [\, \varphi_2(x_j) \vee \varphi_3(x_k)\,] = [\, \varphi_1(x_i) \wedge \varphi_2(x_j)\,] \vee [\, \varphi_1(x_i) \wedge \varphi_3(x_j)\,]$, and $\varphi_1(x_i) \vee [\, \varphi_2(x_j) \wedge \varphi_3(x_k)\,] = [\, \varphi_1(x_i) \vee \varphi_2(x_j)\,] \wedge [\, \varphi_1(x_i) \vee \varphi_3(x_j)\,]$. Let us call $\varphi_4(\cdot)$ the model generated by $\varphi_1(x) \wedge [\varphi_2(x) \vee \varphi_3(x)]$, and $\varphi_5(\cdot)$ the model generated by $[\varphi_1(x) \wedge \varphi_2(x)] \vee [\varphi_1(x_)) \wedge \varphi_3(x_))]$. Our goal is to show that the model for $\varphi_4(\cdot)$ is equivalent to the model for $\varphi_5(\cdot)$. In general, these two models are:

$$m_{\varphi_4}(x) = \min(\alpha_1, \max(\alpha_2, \alpha_3));$$
$$m_{\varphi_4}(\overline{x}) = 1 - \min(\beta_1, \max(\beta_2, \beta_3));$$
$$m_{\varphi_4}(\Theta) = \min(\beta_1, \max(\beta_2, \beta_3)); - \min(\alpha_1, \max(\alpha_2, \alpha_3));\ \text{and}$$

$$m_{\varphi_5}(x) = \max(\min(\alpha_1, \alpha_2), \min(\alpha_1, \alpha_3));$$
$$m_{\varphi_5}(\overline{x}) = 1 - \max(\min(\beta_1, \beta_2), \min(\beta_1, \beta_3));$$
$$m_{\varphi_5}(\Theta) = \max(\min(\beta_1, \beta_2), \min(\beta_1, \beta_3)) - \max(\min(\alpha_1, \alpha_2), \min(\alpha_1, \alpha_3)).$$

Now, consider the focal set x. We have three cases (other possible cases are equivalent to these three after applying the commutativity rule): (a) $\alpha_1 \le \alpha_2 \le \alpha_3$; (b) $\alpha_2 \le \alpha_1 \le \alpha_3$; and (c) $\alpha_2 \le \alpha_3 \le \alpha_1$. The mass associated to the focal set x is:

(a) $m_{\varphi_4}(x) = \alpha_1 = m_{\varphi_5}(x)$;

(b) $m_{\varphi_4}(x) = \alpha_1 = m_{\varphi_5}(x)$; and

(c) $m_{\varphi_4}(x) = \alpha_3 = m_{\varphi_5}(x)$;

i.e., $m_{\varphi_4}(x) = m_{\varphi_5}(x)$ in all the cases. For the focal set $\overline{x}$ we also have three basic cases: (a) $\beta_1 \le \beta_2 \le \beta_3$; (b) $\beta_2 \le \beta_1 \le \beta_3$; and (c) $\beta_2 \le \beta_3 \le \beta_1$; which render:

(a) $m_{\varphi_4}(\overline{x}) = 1 - \beta_1 = m_{\varphi_5}(\overline{x})$;

(b) $m_{\varphi_4}(\overline{x}) = 1 - \beta_1 = m_{\varphi_5}(\overline{x})$; and

(c) $m_{\varphi_4}(\overline{x}) = 1 - \beta_3 = m_{\varphi_5}(\overline{x})$;

Based on the cases above, it can be shown that also $m_{\varphi_4}(\Theta) = m_{\varphi_5}(\Theta)$, proving distributivity for the logical AND operation. A proof for distributivity for the logical OR operation is obtained by following a similar procedure.

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