A Dempster-Shafer Theoretic Conditional Approach to Evidence Updating for Fusion of Hard and Soft Data

Kamal Premaratne, Manohar N. Murthi
Electrical and Computer Engineering
University of Miami
Coral Gables, FL, USA.
{kamal,mmurthi}@miami.edu

Jinsong Zhang
Electrical Engineering
University of Tulsa
Tulsa, OK, USA.
jinsong-zhang@utulsa.edu

Matthias Scheutz
Cognitive Sci., Comp. Sci., and Informatics
Indiana University
Bloomington, IN, USA.
mscheutz@indiana.edu

Peter H. Bauer
Electrical Engineering
University of Notre Dame
Notre Dame, IN, USA.
pbauer@nd.edu

Abstract – *The problem of fusing hard data with soft data is an important issue that has attracted recent attention. In order to provide effective fusion, one must employ an analytical framework that can capture the uncertainty inherent in hard and soft data. In particular, computational linguistic parsing of text-based data results in logical propositions that inherently possess significant semantic ambiguity. A fusion framework must be able to exploit the respective advantages of hard and soft data while mitigating their particular weaknesses. In this paper, we describe a Dempster-Shafer theoretic approach to hard and soft data fusion that relies upon the novel conditional approach of updating. The conditional approach engenders a more flexible method in which one can tune and adapt update strategies. In addition, it provides guidance regarding the order of evidence updating when one takes into account computational complexity constraints. This has important implications in working with models that convert propositional logic statements from text into Dempster-Shafer theoretic form.*

Keywords: Evidence fusion, evidence updating, soft information, Dempster-Shafer theory, conditional approach.

1 Introduction

1.1 Motivation

Suppose various types of sensors are deployed to detect the occurrence of some activity. In a military context, this *activity* could be the transportation of explosive material, ammunition, etc., from one location to another, along with various other related *events* such as the detonation of bombs/improvised explosive devices (IEDs), ambushes, etc., along the route. In a more civilian context, the activity of interest could involve border control and security, emergency response, management of traffic and crowds, etc. The surveillance and the decision making process in such an environment involves gathering data from a variety of disparate physics-based *hard sensors* (e.g., visual, IR, satellite imagery) for identifying different attributes of these events (e.g., location and loudness of an explosion). The raw signal stream generated from each such hard sensor would then undergo signal/noise conditioning/processing so that a 'report' can be generated and sent to a fusion center for fusion with other reports for analysis, information mining, knowledge extraction, object/activity/intent recognition, and the determination of a course of action (CoA).

While the increased availability of a multitude of streaming/stored sensor and data feeds and databases has made the tasks of coordinated, persistent, and pervasive surveillance and decision making easier, it has also been the major contributing factor for the immense burden that intelligence officers are forced to bear. To ease this burden, increased automation of the decision making process is warranted. This, in turn, calls for effective, computationally viable, and analytically tractable models for representation of data and schemes for evidence fusion.

1.2 Challenges

The enormous challenge one faces in achieving this goal is mainly due to the critical role that *soft sensors,* such as COMINT (e.g., intercepted communication chatter), HUMINT (e.g., domain expert knowledge, informant statements, results from interrogations), and OSINT (e.g., information gleaned from mostly unstructured open-source intelligence, such as, newspapers, radio, databases), play in the decision making process. How should the more 'qualitative' information provided by such soft sensors be fused with the more 'quantitative' information generated from the hard sensors? This issue has recently attracted considerable attention from the data and evidence fusion community.

Consider for example a time event chart (TEC) generated from report generated from a hard sensor [1]. A TEC represents events of interest in chronological order and each event is associated with its time of occurrence or the time interval within which the event may have occurred. While a TEC generated from a hard report would typically possess narrower time intervals, a TEC generated from a soft report would typically involve wider time intervals reflecting the more subjective nature of the evidence. See Fig. 1. Indeed, an event time interval in a soft TEC might even precede, or even be 'disjoint' with, the event time interval that a hard TEC provides.

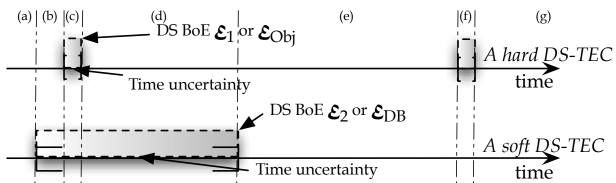

Figure 1: DS-TECs corresponding to hard and soft reports.

Moreover, given that modeling the errors and uncertainties associated with soft information is extremely difficult, or perhaps even impossible [2], how does one fuse the temporally sequenced evidence contained in hard and soft TECs? Being beholden to a-priori assumptions regarding the underlying distributions and priors, the Bayesian framework is unlikely to be successful in this task. For example, the Bayesian recipe of relying on data imputation models to handle missing values may render decisions to be reliable only as long as the assumptions made to impute the missing values reflect reality.

1.3 Contribution

In this paper, we propose a Dempster-Shafer (DS) theoretic evidence updating method that appears to be better suited for fusing hard and soft information. DS theory overcomes Bayesian probability's drawback of a-priori assumptions regarding the underlying distributions and priors to a large extent [3, 4]. To quote [3], *"... for applications with limited a-priori objective knowledge, [DS] evidential reasoning can be superior in the sense that it is better to be only partially but correctly informed than to be completely but incorrectly informed."* DS belief theory, when compared to the probabilistic approach and the possibilistic fuzzy reasoning method, allows for accommodating a wider variety of data imperfections [3, 4]. For example, DS theory offers ways to effectively model *uncertain* implication rules while preserving the material implications of propositional logic statements that they represent [5]; probability theory cannot adequately well capture the evidence of such rules [6]. Additionally, the DS theoretic belief and plausibility measures provide significantly richer information regarding the confidence one may place on the occurrence of an event [7]. So, with a DS theoretic technique, one can arrive at a decision with the full understanding of the associated underlying uncertainties. Also, DS theory allows for a very easy transition to, and from, probability theory [8].

With DS theoretic notions being incorporated, we can utilize a TEC to better reflect the available evidence one has in support of an event. We refer to this as a *DS-TEC.* So, in addition to a TEC's depiction of the time interval associated with each event, a DS-TEC would indicate the DS mass allocated to the event. In Fig. 1, $\mathcal{E}_1$ and $\mathcal{E}_2$ are examples of hard and soft DS-TECs.

This paper is organized as follows: Section 2 provides a brief review of essential DS theoretic notions; Section 3 proposes our new conditional approach to updating; Section 4 is a discussion on how the various fusion parameters could be selected; Section 5 contains an example to illustrate the proposed methodology for fusion of hard and soft evidence; and finally, the concluding remarks appear in Section 6.

2 Preliminaries

2.1 Dempster-Shafer (DS) Theory

2.1.1 Basic Notions

In DS theory, the total set of mutually exclusive and exhaustive propositions of interest is referred to as its *frame of discernment (FoD)* $\Theta = \{\theta_1, \ldots, \theta_n\}$ [9]. A singleton proposition θ_i represents the lowest level of discernible information. Elements in 2^Θ, where 2^Θ denotes the power set of Θ, form all the propositions of interest. We use $\overline{A}$ to denote all singletons in Θ that are not included in A.

Definition 1 *Consider the FoD* Θ *and* $A \subseteq \Theta$.

(i) The mapping $m_\Theta(\centerdot) : 2^\Theta \mapsto [0, 1]$ *is a* basic belief assignment (BBA) *or* mass assignment *if* $m_\Theta(\emptyset) = 0$ *and* $\sum_{A\subseteq\Theta} m_\Theta(A) = 1$. *The BBA is said to be* vacuous *if the only proposition receiving a non-zero mass is* Θ*; the BBA is said to be* Dirichlet *if the only propositions receiving non-zero mass are the singletons and* Θ.

(ii) The belief *of* A *is* $Bl_\Theta(A) = \sum_{B\subseteq A} m_\Theta(B)$.

(iii) The plausibility *of* A *is* $Pl_\Theta(A) = 1 - Bl_\Theta(\overline{A})$. ■

DS theory models the notion of *ignorance* by allowing the mass assigned to a composite proposition to move into its constituent singletons. A proposition that possesses non-zero mass is a *focal element.* The set of focal elements is the *core* $\mathfrak{F}_\Theta$*;* the triple $\{\Theta, \mathfrak{F}_\Theta, m_\Theta(\centerdot)\}$ is the corresponding *body of evidence (BoE).* While $m_\Theta(A)$ measures the support assigned to proposition A *only,* the belief represents the total support that can move into A without any ambiguity; $Pl_\Theta(A)$ represents the extent to which one finds A plausible. When focal elements are constituted of singletons only, belief functions become probability functions. Then, the BBA, belief and plausibility all reduce to probability.

2.1.2 Evidence Combination

Definition 2 (Dempster's Combination Rule (DCR)) *The DCR-fused BoE* $\mathcal{E} \equiv \mathcal{E}_1 \oplus \mathcal{E}_2 = \{\Theta, \mathfrak{F}_\Theta, m_\Theta(\bullet)\}$ *generated from the BoEs* $\mathcal{E}_i = \{\Theta_i, \mathfrak{F}_{\Theta_i}, m_{\Theta_i}(\bullet)\}$ *with* $\Theta \equiv \Theta_i$*, where* $i = 1, 2$*, is*

$$m_\Theta(A) = \sum_{C \cap D = A} m_{\Theta_1}(C)\, m_{\Theta_2}(D)/(1-K),\ \forall A \subseteq \Theta,$$

whenever $K = \sum_{C\cap D=\emptyset} m_{\Theta_1}(C)\, m_{\Theta_2}(D) \neq 1.$ ■

Note that $K \in [0, 1]$. A higher K value indicates more conflict between evidence provided by the BoEs; a lower K value indicates more agreement between the BoEs. Hence, K is referred to as the *conflict* between the BoEs being fused. The DCR's difficulties in fusing conflicting BoEs are well documented in the literature. The requirement that the two FoDs being fused be identical constitutes another drawback associated with the DCR.

To fuse evidence generated from non-exhaustive FoDs Θ_1 and Θ_2 (so that $\Theta_1 \neq \Theta_2$ and $\Theta_1 \cap \Theta_2 \neq \emptyset$), one can simply ignore the differences in the FoDs by having each source allocate zero mass to propositions that are not within its own FoD and continue applying DCR. In essence, this approach assumes that each source can discern $\Theta_1 \cup \Theta_2$ and ignores the fact that some propositions are not within its scope of expertise. The counter-intuitive conclusions this approach may generate are well documented [10].

In the deconditioning approach, each source would artificially introduce ambiguities into its evidence so that its own FoD is 'deconditioned' or 'expanded' to $\Theta_1 \cup \Theta_2$. For example, consider the *plausibilities correction method (PCM)* in [10]: Let $\Theta_C = \Theta_1 \cap \Theta_2$. Then the propositions of $(\Theta_1 \cap \overline{\Theta}_C)$ are discerned by the first source alone, those of $(\Theta_2 \cap \overline{\Theta}_C)$ are discerned by the second source alone, and the propositions of Θ_C are discerned by both. A deconditioning step is applied to the partial knowledge of sources to 'refer' their knowledge to $\Theta_1 \cup \Theta_2$. In the PCM, combination is performed via the multiplication of these 'deconditioned' plausibilities; it requires that the plausibilities of only singleton propositions are maintained throughout. Hence, after combination, it is impossible to obtain a valid BBA and information for any composite proposition.

2.2 Conditional Approach to Updating

The conditional approach to fusing evidence is based on the premise that one has to 'condition' or 'update' the already available evidence with respect to what both FoDs can discern [11, 12]. In this sense, it is more appropriate to refer to it as an evidence *update* strategy than an evidence *combination* strategy. Once the conditioning operation is performed, each source invokes a strategy to incorporate its originally cast evidence that does not belong to Θ_C. This approach enables a source to update its own knowledge base, and exchange information with other sources for the express purpose of refining its own knowledge, without having to continually 'expand' its FoD.

2.2.1 Conditional Update Strategy

Restricting ourselves to the identical FoD case, let us explain the conditional update strategy in [11]; its extension to the more general non-exhaustive FoD case appears in [12]. Consider the two BoEs $\mathcal{E}_1$ and $\mathcal{E}_2$. We identify them to be at state k, $k \geq 0$, (after all, we are dealing with an *update* strategy) via $\mathcal{E}_i[k] = \langle \Theta_i, \mathfrak{F}_{\Theta_i}[k], m_{\Theta_i}(\bullet)[k]\rangle$, where $\Theta \equiv \Theta_i$, for $i = 1, 2$. The evidence update strategy in [11] that updates the evidence in $\mathcal{E}_1[k]$ with the evidence available in $\mathcal{E}_2[k]$ to yield the updated BoE $\mathcal{E}_1[k+1] = \langle \Theta_1, \mathfrak{F}_{\Theta_1}[k+1], m_{\Theta_1}(\bullet)[k+1]\rangle \equiv \mathcal{E}_1[k] \lhd \mathcal{E}_2[k]$, $k \geq 0$, is

$$\begin{aligned} &Bl_{\Theta_1}(B)[k+1] \\ &\quad = \alpha(A)[k]\, Bl_{\Theta_1}(B)[k] + \beta(A)[k]\, Bl_{\Theta_2}(B|A)[k], \quad (1) \end{aligned}$$

where $Bl_{\Theta_2}(A) > 0$. The parameters $\{\alpha(A)[\bullet], \beta(A)[\bullet]\}$ are non-negative, functions of the conditioning event A only, and satisfy $\alpha(A)[\bullet] + \beta(A)[\bullet] = 1$. Similar update equations exist for the mass and plausibility updates [11].

2.2.2 DS Theoretic Conditionals

In [11, 12], the conditional operation in (1) is implemented using the Fagin-Halpern (FH) DS theoretic conditionals.

Definition 3 *[13] For* $\mathcal{E} = \{\Theta, \mathfrak{F}_\Theta, m_\Theta(\bullet)\}$*,* $A, B \subseteq \Theta$ *with* $Bl_\Theta(A) > 0$*, the* conditional belief *of* B *given* A *is*

$$Bl_\Theta(B|A) = Bl_\Theta(A \cap B)/[Bl_\Theta(A \cap B) + Pl_\Theta(A \cap \overline{B})].$$

A similar equation yields the conditional plausibility $Pl_\Theta(B|A)$. ■

The conditioning event A identifies the event of 'occurrence' leading to the change of 'status' that warrants an update. Although different notions of DS theoretic conditional notions are available and used in the literature, regarding the FH conditionals and its use in the update strategy in (1), we make the following observations: • Because $Bl_\Theta(B|A) = Bl_\Theta(A \cap B|A)$, while evaluating the evidence we have in support of B when our view is restricted to only A, the FH conditionals consider only those propositions that both A and B have in common. • The FH conditionals provide a more appropriate probabilistic interpretation and a more natural and fluid transition to Bayesian notions [11, 13]. Indeed, it is the FH conditional belief and plausibility that correspond precisely to the inner and outer measures of a nonmeasurable event (for which a probability has not been assigned) [14]. See [11] for a detailed interpretation. • With the FH conditionals, (1) has an interesting probabilistic interpretation in the limiting case when the focal elements are singletons only [11]. • The conditional update strategy in (1) is more robust than the DCR [12] in the sense that the sensitivity of the fused BoE's masses are less affected by perturbations in the masses of the BoEs being fused. • The conditional update strategy in (1) forms the basis of *evidence filtering,* a novel technique that exploits DS notions and digital filtering techniques for detecting multimodal faint signatures possibly buried in clutter [15]. Its use in evidence filtering clearly demonstrates the effectiveness of the conditional update strategy in (1) when dealing with temporally sequenced data.

3 A Conditional Approach for Fusing Hard and Soft Temporal Data

With these attractive properties, conditional updating appears to be ideally suited for integrating the temporally sequenced information contained in hard and soft DS-TECs.

3.1 Conditional Update Equation (CUE)

To update one DS-TEC with the evidence received from another DS-TEC however requires an update strategy that can handle new evidence received in the form of a general BoE. However, the strategy in (1) is applicable only to the case when a source is being updated with the 'occurrence' of one proposition A, i.e., the incoming evidence has *only one* focal element! We therefore propose the following generalization:

Definition 4 (Conditional Update Equation (CUE)) *For the BoEs $\mathcal{E}_i[k]$ with $\Theta \equiv \Theta_1 = \Theta_2$, the* conditional update equation (CUE) *that updates $\mathcal{E}_1[k]$ with the evidence in $\mathcal{E}_2[k]$ is $\mathcal{E}_1[k+1] \equiv \mathcal{E}_1[k] \lhd \mathcal{E}_2[k]$, $\forall k \geq 0$, where*

$$Bl_{\Theta_1}(B)[k+1] = \alpha[k]\, Bl_{\Theta_1}(B)[k] + \sum_{A \subseteq \Theta_2} \beta(A)[k]\, Bl_{\Theta_2}(B|A)[k].$$

The CUE parameters $\{\alpha[\centerdot], \beta(A)[\centerdot]\}$ are non-negative and satisfy $\alpha[\centerdot] + \sum_{A\subseteq\Theta_2} \beta(A)[\centerdot] = 1$, where $\beta(A)[\centerdot] = 0, \forall A \notin \mathfrak{F}_{\Theta_2}[\centerdot]$, i.e., $\alpha[\centerdot] + \sum_{A\in\mathfrak{F}_{\Theta_2}[\centerdot]} \beta(A)[\centerdot] = 1$. ■

Remarks:

1. Note that, $Bl_{\Theta_1}(\emptyset)[\centerdot] = 0$ and $Bl_{\Theta_1}(\Theta_1)[\centerdot] = 1$.

2. $Bl_{\Theta_1}(B)[k]$ (computed in $\mathcal{E}_1[k]$) accounts for the evidence that $\mathcal{E}_1$ already has for $B \subseteq \Theta_1$; $Bl_{\Theta_2}(B|A)[k]$ (computed in $\mathcal{E}_2[k]$) accounts for the evidence provided by $A \in \mathfrak{F}_{\Theta_2}$ in $\mathcal{E}_2$.

3. The corresponding mass update equation is

$$m_{\Theta_1}(B)[k+1] = \alpha[k]\, m_{\Theta_1}(B)[k] + \sum_{A \subseteq \Theta_2} \beta(A)[k]\, m_{\Theta_2}(B|A)[k]. \quad (2)$$

Noting that $m_{\Theta_2}(B|A) = 0$ if $\overline{A} \cap B \neq 0$ [11], we get the mass update of Θ_1 as

$$m_{\Theta_1}(\Theta_1)[k+1] = \alpha[k]\, m_{\Theta_1}(\Theta_1)[k] + \beta(\Theta_2)[k]\, m_{\Theta_2}(\Theta_2)[k]. \quad (3)$$

3.2 Some Properties of the CUE

3.2.1 Basic Properties

The following can be easily established.

Claim 1 *Let $\mathcal{E}_\Theta$ denote the vacuous BoE.*

(i) $\mathcal{E}_\Theta \lhd \mathcal{E}_\Theta = \mathcal{E}_\Theta$ for arbitrary CUE parameters; $\mathcal{E}_1 \lhd \mathcal{E}_1 = \mathcal{E}_1$ when $\beta(A)[\centerdot] = 0, \forall A \subset \Theta_2$.

(ii) For $\mathcal{E}_1[1] = \mathcal{E}_1 \lhd \mathcal{E}_\Theta$, $\mathcal{E}_1 \neq \mathcal{E}_\Theta$, $m_{\Theta_1}(\Theta_1)[1] > m_{\Theta_1}(\Theta_1)$ and, if $\alpha < 1$, $m_{\Theta_1}(B)[1] < m_{\Theta_1}(B)$, $\forall B \in \mathfrak{F}_{\Theta_1} \setminus \Theta_1$.

(iii) For $\mathcal{E}_1[1] = \mathcal{E}_\Theta \lhd \mathcal{E}_2$, $\mathcal{E}_2 \neq \mathcal{E}_\Theta$, $m_{\Theta_1}(B)[1] \geq m_{\Theta_1}(B)$, $\forall B \subset \Theta_1$, and, if $\alpha < 1$, $m_{\Theta_1}(\Theta_1)[1] < m_{\Theta_1}(\Theta_1) = 1$. ■

A result that may prove useful as guidance on how to select the CUE parameters is

Claim 2

(i) Updated mass of a given proposition increases, i.e., $m_{\Theta_1}(B)[k+1] > m_{\Theta_1}(B)[k]$, $B \subseteq \Theta_1$, iff

$$\sum_{A\subseteq\Theta_2} \beta(A)[k]\, m_{\Theta_2}(B|A)[k] > m_{\Theta_1}(B)[k] \sum_{A\subseteq\Theta_2} \beta(A)[k].$$

(ii) Updated mass of the completely ambiguous proposition decreases, i.e., $m_{\Theta_1}(\Theta_1)[k+1] < m_{\Theta_1}(\Theta_1)[k]$, iff

$$\beta(\Theta_2)[k]\, m_{\Theta_2}(\Theta_2)[k] < m_{\Theta_1}(\Theta_1)[k] \sum_{A\subseteq\Theta_2} \beta(A)[k].$$

■

3.2.2 Focal Elements

Note that, the CUE may generate focal elements other than those in $\mathfrak{F}_{\Theta_1}[k] \cup \mathfrak{F}_{\Theta_2}[k]$ because $m_{\Theta_2}(B) = 0$ does not guarantee $m_{\Theta_2}(B|A) = 0$ (see [11] for details). Therefore, $B \in \mathfrak{F}_{\Theta_1}[k+1]$ implies that $B \in \mathfrak{F}_{\Theta_1}[k]$ or $B \subseteq A$ for some $A \in \mathfrak{F}_{\Theta_2}[k]$ for which $\beta(A)[k] \neq 0$.

3.2.3 Mutual Updating

With no additional external evidence sources, CUE can be used to mutually constrain the interpretation of two sources by recursively updating each BoE from the evidence of the other. If each BoE converges to a limiting BoE, it would indicate an 'agreement' between the two BoEs as to the extent to which each can benefit from the other.

To study this, let $\{\alpha[k], \beta(\centerdot)[k]\}$ and $\{\gamma[k], \delta(\centerdot)[k]\}$ be the CUE parameters corresponding to $\mathcal{E}_1[k+1] = \mathcal{E}_1[k] \lhd \mathcal{E}_2[k]$ and $\mathcal{E}_2[k+1] = \mathcal{E}_1[k] \rhd \mathcal{E}_2[k]$, respectively. Then, the masses of $\mathcal{E}_1[k+1]$ and $\mathcal{E}_2[k+1]$ can be expressed via the following matrix recursions:

$$\begin{aligned} \mathbf{m}(B)[k+1] &= \Gamma[k]\, \mathbf{m}(B)[k] + \Phi(B)[k], \forall B \subset \Theta; \\ \mathbf{m}(\Theta)[k+1] &= \Gamma[k]\, \mathbf{m}(\Theta)[k], \end{aligned} \quad (4)$$

where

$$\mathbf{m}(B)[\centerdot] = \begin{bmatrix} m_{\Theta_1}(B)[\centerdot] \\ m_{\Theta_2}(B)[\centerdot] \end{bmatrix}; \; \mathbf{m}(\Theta)[\centerdot] = \begin{bmatrix} m_{\Theta_1}(\Theta)[\centerdot] \\ m_{\Theta_2}(\Theta)[\centerdot] \end{bmatrix};$$

$$\Gamma[\centerdot] = \begin{bmatrix} \alpha[\centerdot] & \beta(\Theta_2)[\centerdot] \\ \delta(\Theta_1)[\centerdot] & \gamma[\centerdot] \end{bmatrix};$$

$$\Phi(B)[\centerdot] = \begin{bmatrix} \sum_{A\subset\Theta_2} \beta(A)[k]\, m_{\Theta_2}(B|A)[k] \\ \sum_{A\subset\Theta_1} \delta(A)[k]\, m_{\Theta_1}(B|A)[k] \end{bmatrix}. \quad (5)$$

Note that the CUE parameters are non-negative and satisfy

$$\alpha[\centerdot] + \sum_{A \subseteq \Theta_2} \beta(A)[\centerdot] = \gamma[\centerdot] + \sum_{A \subseteq \Theta_1} \delta(A)[\centerdot] = 1, \quad (6)$$

where $\beta(A)[\centerdot] = 0, \forall A \notin \mathfrak{F}_{\Theta_2}[\centerdot]$, and $\delta(A)[\centerdot] = 0, \forall A \notin \mathfrak{F}_{\Theta_1}[\centerdot]$. The next result indicates that with mutual updating, each BoE can indeed reduce its ignorance.

Lemma 3 (General BoEs) *Consider the CUE-based mutual updates $\mathcal{E}_1[k+1] = \mathcal{E}_1[k] \lhd \mathcal{E}_2[k]$ and $\mathcal{E}_2[k+1] = \mathcal{E}_1[k] \rhd \mathcal{E}_2[k]$, for $k \geq 0$, in* (4). *If $\alpha[k] + \beta(\Theta_2)[k] \leq \rho < 1$ and $\gamma[k] + \delta(\Theta_1)[k] \leq \rho < 1$, for all $k \geq 0$, the mass for the completely ambiguous proposition vanishes in the limit, i.e., $m^*_{\Theta_1}(\Theta_1) = m^*_{\Theta_2}(\Theta_2) = 0$.* □

Proof: Obvious because $\|\Gamma[k]\|_\infty \leq \rho < 1, \forall k$. ■

Dirichlet BoEs: In this case, (6) reduces to

$$\begin{aligned} 1 &= \alpha[\centerdot] + \sum_{B \in \Theta_2} \beta(B)[\centerdot] + \beta(\Theta_2)[\centerdot] \\ &= \gamma[\centerdot] + \sum_{B \in \Theta_1} \delta(B)[\centerdot] + \delta(\Theta_1)[\centerdot]. \end{aligned} \quad (7)$$

Accordingly, the matrix $\Phi(B)[\centerdot]$ in (4) reduces to

$$\Phi(B)[k] = \begin{bmatrix} \beta(B)[\centerdot] \\ \delta(B)[\centerdot] \end{bmatrix}, \ \forall B \in \Theta. \quad (8)$$

Lemma 4 (Dirichlet BoEs) *Consider the CUE-based mutual updates $\mathcal{E}_1[k+1] = \mathcal{E}_1[k] \lhd \mathcal{E}_2[k]$ and $\mathcal{E}_2[k+1] = \mathcal{E}_1[k] \rhd \mathcal{E}_2[k]$, for $k \geq 0$, with $\mathcal{E}_i[0]$, $i = 1, 2$, being Dirichlet. Then the following are true:*

(i) $\mathcal{E}_i[\centerdot]$, $i = 1, 2$, are Dirichlet.

(ii) If all the CUE parameters are time-invariant, the limiting BoEs $\mathcal{E}^*_i$, $i = 1, 2$, exist and are purely probabilistic; the limiting masses of the singletons are

$$\mathbf{m}^*(B) \equiv \begin{bmatrix} m^*_{\Theta_1}(\Theta_1) \\ m^*_{\Theta_2}(\Theta_2) \end{bmatrix} = \frac{1}{M} \begin{bmatrix} 1-\gamma & \beta(\Theta_2) \\ \delta(\Theta_1) & 1-\alpha \end{bmatrix} \begin{bmatrix} \beta(B) \\ \delta(B) \end{bmatrix},$$

where $M \equiv (1-\alpha)(1-\gamma) - \beta(\Theta_2)\,\delta(\Theta_1)$. □

Proof:

(i) This follows from the observations in Section 3.2.2.

(ii) Apply z-transform to the update equations and solve for $m_{\Theta_i}(B)[\centerdot]$, $i = 1, 2$. The Final Value Theorem then yields the results being claimed. ■

Remark: The expressions in item (ii) above can be used to arrive at expressions for the limiting masses in the 'extreme' cases. For example, $\lim_{\beta(B) \to 0} m^*_{\Theta_1}(B) = \dfrac{\beta(\Theta_2)\,\delta(B)}{M}$; $\lim_{\beta(B) \to 1-\alpha} m^*_{\Theta_1}(B) = 1$; etc.

3.2.4 Repeated Updating

Suppose $\mathcal{E}_1$ is repeatedly conditioned by the same BoE $\mathcal{E}_2$, i.e., $\mathcal{E}_1[k+1] = \mathcal{E}_1[k] \lhd \mathcal{E}_2$, $k \geq 0$. With $\mathcal{E}_2$ showing little or no change (e.g., a faulty/compromised sensor), does $\mathcal{E}_1$ continue to update itself? Applying the CUE repeatedly, and assuming $\alpha[\centerdot] = \alpha < 1$, $\beta(A)[\centerdot] = \beta(A)$, and a static $\mathcal{E}_2$, we get the limiting value of $Bl_{\Theta_1}(B)[k]$ as

$$Bl^*_{\Theta_1}(B) = \sum_{A \in \mathfrak{F}_{\Theta_2}} [\beta(A)/(1-\alpha)]\, Bl_{\Theta_2}(B|A). \quad (9)$$

So, the information that $\mathcal{E}_1$ had would eventually be completely eroded in favor of information that is fully determined by $\mathcal{E}_2$, an undesirable situation if $\mathcal{E}_2$ is not reliable.

4 Tuning and Adapting Strategies

4.1 Selection of Parameters

4.1.1 Selection of $\alpha[\centerdot]$

A higher $\alpha[\centerdot]$ can capture the inflexibility of available evidence towards changes, perhaps because of its perception of the low reliability of the incoming evidence and/or high inertia of the available evidence (e.g., when it is reluctant to change knowledge that has been already gathered from a vast collection of reliable evidence). A lower $\alpha[\centerdot]$ can capture flexibility towards changes, perceived reliability of incoming evidence, or towards the initial phase of evidence collection when the BoE has little or no credible knowledge base to begin with. Another strategy is to give each 'piece' of already gathered evidence and the incoming new evidence equal 'importance'. These notions give rise to

Definition 5 (Inertia-Based Selection) *With reference to the CUE, (i)* infinite inertia-based updating *refers to $\alpha[k] = 1$; (ii)* zero inertia-based updating *refers to $\alpha[k] = 0$; (iii)* proportional inertia-based updating *refers to $\alpha[k] = N/(N+1)$, where N is the number of 'pieces' of evidence on which the available evidence is based upon.* ■

4.1.2 Selection of $\beta(A)[\centerdot]$

Here, we immediately see two choices.

Definition 6 *The choice of $\beta(\centerdot)[\centerdot]$ yields two CUE versions.*

(i) CUE_ext *refers to $\beta(A)[\centerdot] = K_{\Theta_2} m_{\Theta_2}(A)[\centerdot]$, $\forall A \in \mathfrak{F}_{\Theta_2}[\centerdot]$, where K_{Θ_2} is a constant.*

(ii) CUE_int *refers to $\beta(A)[\centerdot] = K_{\Theta_1} m_{\Theta_1}(A)[\centerdot]$, $\forall A \in \mathfrak{F}_{\Theta_2}[\centerdot] \cap \mathfrak{F}_{\Theta_1}[\centerdot]$ and $\beta(A)[\centerdot] = 0$, $\forall A \in \mathfrak{F}_{\Theta_2}[\centerdot] \setminus \mathfrak{F}_{\Theta_1}[\centerdot]$, where K_{Θ_1} is a constant.* ■

Remarks:

1. The CUE_ext strategy 'weighs' the incoming evidence according to the support each focal element receives from $\mathcal{E}_2$. The CUE_ext strategy has an interesting Bayesian interpretation: it reduces to a weighted average of the probability mass functions corresponding to $\mathcal{E}_1$ and $\mathcal{E}_2$.

2. The CUE_int strategy 'weighs' the incoming evidence according to the support each focal elements receives from $\mathcal{E}_1$. So the focal elements of the updated BoE are restricted to within $\mathfrak{F}_{\Theta_1}$.

3. One can show that, with Dirichlet BoEs, both CUE_ext and CUE_int strategies ensure convergence with mutual updating (even when the CUE parameters are time varying).

5 Incorporating Soft Evidence

5.1 An Example

5.1.1 Setup

Consider a surveillance/monitoring scenario where 'objects' crossing a security zone perimeter are to be classified.

Hard Evidence: Various hard sensors can be used to classify objects as they approach and cross the security zone perimeter. For instance, using the facilities available at the MObile SEnsor Systems (MOSES) Laboratory at the University of Notre Dame, we have actually utilized a scheme where the modalities $ACC = Acoustic$, $PIR = Passive\_IR$, $RSSI = Received\_Signal\_Strength\_Indicator$, and $VIB = Vibration$ are used for this purpose. A simple threshold identifies each sensor signal as either $1 = High$ (activated) or $0 = Low$ (not activated), and activation of each modality is used to classify the objects into the three classes in $\Theta_{Obj} = \{P = Person, V = Vehicle, O = Other\}$ according to an appropriate mapping. To illustrate the CUE-based evidence update scheme, let us consider the following BoE to reflect the confidence we place on the object being a person when both the PIR and ACC sensor modalities are activated:

$$\mathcal{E}_{Obj} : \; m_{Obj}(P) = 0.8; \; m_{Obj}(\Theta_{Obj}) = 0.2. \qquad (10)$$

Each person (P) or vehicle (V) may belong to friendly forces or an enemy/terrorist group, thus generating the subcategories $\{P_F, P_E\}$ and $\{V_F, V_E\}$; O could account for an object, such as an animal (e.g., a rabbit, dog), that cannot be further sub-classified.

Soft Evidence: Hard sensor information cannot be used to further refine the objects $\{P, V\}$ into their sub-categories; soft information would be essential for this purpose. For example, the prevailing threat level TL in the proximity of the security zone may provide evidence on how we may view, and further refine, each perimeter crossing. To assess the prevailing TL, we may search databases for information containing insurgency attacks in the region. We can limit the search to a given time period via purely standard methods using temporal indices, keyword search, etc. All applicable sentences (e.g., sentences that might have potentially relevant information because they contain the right keywords and they had been indexed for the right time frame, etc.) can be parsed using a method for automatically translating natural language (NL) expressions into a logical form (using formal logics in conjunction with categorical grammar and lambda calculus); details of an appropriate translation algorithm is in [16]. This results in (lambda-free) logical expressions that can be used to assign DS theoretic masses.

Specifically, suppose that we are interested in determining the overall threat level TL based on information from the database. Database information can be thought of as implications of the form $\phi \rightarrow_\tau \psi$, where ϕ is a proposition in the database, ψ is the corresponding assertion about the threat level (e.g., $TL = high$) and $\tau \geq 0$ is the certainty of the rule (i.e., the degree to which we are confident that this implication holds). Such implication rules can potentially serve as the means to 'strengthen' or 'refine' the evidence in $\mathcal{E}_{Obj}$. For example, suppose the information extracted from the database implies a high threat level with $\tau = 0.6$. Defined on the FoD $\Theta_{DB} = \{E = Enemy, F = Friend\}$, this can correspond to the BoE

$$\mathcal{E}_{DB} : \; m_{DB}(E) = 0.6; \; m_{DB}(\Theta_{DB}) = 0.4. \qquad (11)$$

In addition to using database information, we can also use soft information obtained from interviewing suspects. For example, an unknown person caught within a secure perimeter might claim that they are on an authorized mission (e.g., belonging to a repair team that arrived to fix the plumbing). From questioning the person about their reason to be on the base, we might be able to determine whether s/he is indeed an authorized person or not, and use the same method as above to determine the FoD $\Theta_{Int} = \{AM(H)_{low}, AM(H)_{med}, AM(H)_{high}\}$ for an "authorized mission" AM for a human H. As with the database case, we can assign a mass function for assigning masses to the propositions in Θ_{Int} based on the assessment of the interviewer (i.e., the degree to which the interviewer believes that the mission of person H is authorized).

FoD Consolidation and DS-TEC Generation: The CUE in Definition 4 requires identical FoDs. So we employ the strategy of 'extending' the hard and soft BoEs in (10) and (11) to the 'cross-product' FoD $\Theta = \{P_F, P_E, V_F, V_E, O\}$. The BoEs thus generated from $\mathcal{E}_{Obj}$ and $\mathcal{E}_{DB}$ (assuming that $\mathcal{E}_{DB}$'s 'expertise' does not contain O) appear in Table 1 as $\widehat{\mathcal{E}}_{Obj}[0]$ and $\widehat{\mathcal{E}}_{DB}[0]$, respectively. The hard and soft DS-TECs corresponding to $\widehat{\mathcal{E}}_{Obj}[0]$ and $\widehat{\mathcal{E}}_{DB}[0]$ would look very similar to those in Fig. 1: the hard DS-TEC of $\widehat{\mathcal{E}}_{Obj}[0]$ would have a much narrower 'activated' time interval; the time interval of the soft DS-TEC of $\widehat{\mathcal{E}}_{DB}[0]$ would be much wider reflecting the prevailing threat level.

5.1.2 Fusion

One significant advantage of the CUE is that it is easily adaptable to fusion of temporally sequenced, spatially distributed, multi-modal data. Indeed, in the work being conducted at the MOSES Laboratory with *purely hard* sensor data, we have experimented with a conditional update strategy that uses evidence generated from temporally sequenced data from the sensor nodes that are located next to the node being updated.

How should we utilize the CUE to fuse the evidence in *hard and soft* DS-TECs when they may not fully agree on the time of occurrence of an event? The most natural strategy would be to first 'partition' the time axis so that, within each partition, the DS-TECs agree as to the occurrence (or non-occurrence) of an event, and then to carry out fusion within each partition. For example, the hard DS-TEC $\mathcal{E}_1$ and the soft DS-TEC $\mathcal{E}_2$ in Fig. 1 can be fused within the partitions (a)-(g) to get the update $\mathcal{E}_1 \lhd \mathcal{E}_2$. Applying the DCR within each partition may lead to difficulties because

Table 1: Fusion Results for Several Cases. *Note:* $P = (P_F, P_E)$, $E = (P_E, V_E)$, $\overline{O} = (P_F, P_E, V_F, V_E)$.

Propo-sition	***Hard*** $\widehat{\mathcal{E}}_{Obj}[0]$	***Soft*** $\widehat{\mathcal{E}}_{DB}[0]$	***Hard*** $\lhd$ ***Soft*** ***Obj-g***	***Obj-1***	***Obj-2***	***Hard*** $\rhd$ ***Soft*** ***DB-g***	***DB-1***	***DB-2***	***Mutual*** ***M-1***	***M-2***	***DCR***
P_E											0.48
P	0.8		$0.8\,\alpha$	0.53	0.27	$\delta(P) + 0.8\,\delta(\Theta)$	0.30	0.67	0.40	0.69	0.32
E		0.6	$\beta(E) + 0.6\,\beta(\overline{O})$	0.27	0.66	$0.6\,\gamma$	0.40	0.20	0.60	0.31	0.12
$\overline{O}$		0.4	$0.4\,\beta(\overline{O})$	0.07	0	$0.4\,\gamma$	0.27	0.13			0.08
Θ	0.2		$0.2\,\alpha$	0.13	0.07	$0.2\,\delta(\Theta)$	0.03	0			

the BoEs being encountered can be significantly, or even completely, conflicting especially within non-overlapping time intervals. The DCR's difficulties in fusing conflicting BoEs are of course well documented. The CUE does not encounter this problem because of the conditioning operation it first invokes.

Table 2: CUE Parameters in Table 1.

Case	α	$\beta(E)$	$\beta(\overline{O})$	γ	$\delta(P)$	$\delta(\Theta)$
Obj-g	$\alpha+\beta(E)+\beta(\overline{O})=1$			$\gamma+\delta(P)+\delta(\Theta)=1$		
Obj-1	$2/3$	$1/6$	$1/6$			
Obj-2	$1/3$	$2/3$	0			
DB-g	$\alpha+\beta(E)+\beta(\overline{O})=1$			$\gamma+\delta(P)+\delta(\Theta)=1$		
DB-1				$2/3$	$1/6$	$1/6$
DB-2				$1/3$	$2/3$	0
M-1	CUE_ext scheme with $\alpha = 1/3$, $\gamma = 2/3$					
M-2	CUE_ext scheme with $\alpha = 2/3$, $\gamma = 1/3$					

To illustrate the newly proposed CUE-based evidence updating mechanism, we update each DS-TEC from the evidence available during the time partition where $\widehat{\mathcal{E}}_{Obj}$ is activated. The fusion results for several cases appear in Table 1; the CUE parameters selected are in Table 2. (a) **Hard** $\lhd$ **Soft** refers to $\widehat{\mathcal{E}}_{Obj}[1] = \widehat{\mathcal{E}}_{Obj}[0] \lhd \widehat{\mathcal{E}}_{DB}[0]$. (b) **Hard** $\rhd$ **Soft** refers to $\widehat{\mathcal{E}}_{DB}[1] = \widehat{\mathcal{E}}_{Obj}[0] \rhd \widehat{\mathcal{E}}_{DB}[0]$. (c) **Mutual** refers to the mutual updates $\widehat{\mathcal{E}}_{Obj}[k+1] = \widehat{\mathcal{E}}_{Obj}[k] \lhd \widehat{\mathcal{E}}_{DB}[k]$ and $\widehat{\mathcal{E}}_{DB}[k+1] = \widehat{\mathcal{E}}_{Obj}[k] \rhd \widehat{\mathcal{E}}_{DB}[k]$ as $k \to \infty$. Since $\widehat{\mathcal{E}}_{Obj}[0]$ and $\widehat{\mathcal{E}}_{DB}[0]$ have no common focal elements, we implemented the CUE_ext scheme only. For ***M-1***, Fig. 2 shows the focal elements of $\widehat{\mathcal{E}}_{Obj}[k]$, $k = \overline{0, 10}$. (d) **DCR** refers to $\widehat{\mathcal{E}}_{Obj}[0] \oplus \widehat{\mathcal{E}}_{DB}[0]$ in Definition 2.

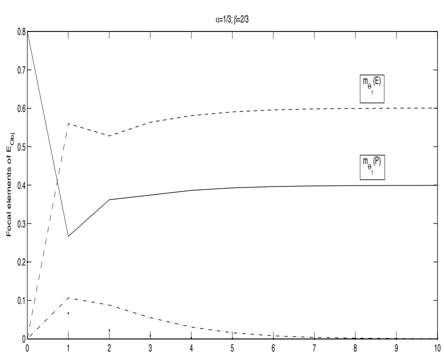

Figure 2: Converging focal elements of $\widehat{\mathcal{E}}_{Obj}[k]$ for ***M-1***.

Remarks:

1. CUE-based schemes can easily adapt to 'highlight' various propositions of interest. For instance, to emphasize E, $\widehat{\mathcal{E}}_{Obj}$ in ***Obj-2*** selects $\beta(\overline{O}) = 0$; to emphasize P, $\widehat{\mathcal{E}}_{DB}$ in ***DB-2*** selects $\delta(\Theta) = 0$.

2. With CUE_ext mutual updating, both sources converge to the same BoE. With $\alpha = 1/3$, $\widehat{\mathcal{E}}_{Obj}$ in ***M-1*** is more receptive to the evidence of $\widehat{\mathcal{E}}_{DB}$ and therefore the limiting BoE has more support for E. The opposite is true in ***M-2***.

3. While the DCR is not too amenable to handle sequenced data, the conditional approach can easily be modified to handle temporally sequenced data [15]. Moreover, note how the DCR tends to allocate significantly higher support for P_E. This is true even if the DCR scheme is implemented with BoE 'discounting' [9]. Usually, soft evidence serves its purpose best when, in light of the prevailing ambient conditions (e.g., threat level), it is used to 'highlight' certain propositions that the hard evidence supports. In this sense, CUE-based schemes appear to be better suited for the current purpose.

5.2 Evidence Ordering: Some Remarks

The 'direction' of updating (i.e., Hard $\lhd$ Soft or Hard $\rhd$ Soft) would often depend on the types of evidence available, how they are generated, how they are made available and how they are to be utilized. Hard $\lhd$ Soft fusion would be useful when soft evidence received from intelligence/pre-mission briefings, real-time intelligence updates, etc., warrants the strengthening or refining of hard evidence being received. On the other hand, Hard $\rhd$ Soft fusion would be useful when a high-level intelligence center may want to exploit real-time physical sensor readings being received to change the tone of intelligence briefings and statements regarding the threat situation.

Computational complexity concerns can also dictate the direction of updating. This is especially true with a CUE-based strategy because of the high computational burden DS theoretic methods entail. One solution is to exploit the CUE's ability to 'highlight' only those propositions of critical interest. Another popular strategy is to use Dirichlet BoEs so that CUE-based (and also DCR-based) fusion results remain Dirichlet. On the other hand, soft information is more likely to be represented in terms of uncertain implication rules of the type $A \implies B$ with a confidence in $[\alpha, \beta]$, where $0 \leq \alpha \leq \beta \leq 1$ [5]. As we previously mentioned, probability theory cannot adequately well capture the evidence contained in such rules [6]. DS theory offers ways to effectively model these implication rules while preserving the material implications of propositional logic statements that such rules represent, viz., reflexivity, transitivity, and contra-positivity [5]. Dirichlet BoEs however are inadequate to model these DS theoretic models. One solution is to use the CUE_int parameter selection strategy to contain the growth of focal elements. Actually, as it turns out, with CUE_int, a Dirichlet hard DS-TEC will remain Dirichlet when it is updated with an arbitrary soft DS-TEC.

6 Concluding Remarks

As its various properties demonstrate, the CUE appears to be ideally suited for fusion of hard and soft evidence. While its DS theoretic basis enables a smooth transition to probability, the CUE also has the ability to smoothly transition between the different BoEs that are being fused (e.g., see the $\alpha = 1$ and $\beta(\overline{O}) = 1$ cases of ***Obj-g*** in Table 1). The CUE also can easily 'highlight' selected propositions (e.g., see ***Obj-2*** and ***DB-2*** in Table 1) so that the tone of the fused result could accordingly be changed. The existence of convergent solutions for the mutual updating schemes means that one is guaranteed to eventually arrive to a point where no further benefit can be extracted from each other: CUE_ext guarantees complete agreement (see ***M-1*** and ***M-2*** in Table 1); CUE_int guarantees an 'agree-to-disagree' status (the complete proofs of these results are not included here).

Previous work amply demonstrates the utility of DS theoretic conditional updating schemes in applications involving temporally sequenced data [15], knowledge extraction from imperfect data, graphical dependency models, etc. CUE clearly holds the same promise. The extension of the CUE to handle non-exhaustive FoDs is also of importance and we are currently looking into this issue.

The DS-theoretic BoE generated by the proposed scheme can be used by a decision-maker (DM) for *decision making under uncertainty,* which is a combination of decision making under *risk* and *ignorance* [17]. "Risk" here means that the DM only knows a probability distribution over the states of nature, while "ignorance" means that no knowledge about the states of nature is available. While in the former, actions with the maximum expected utility can be chosen, in the latter expected values cannot be computed without any additional assumptions that reflect the DM's attitude. For example, the DM could weigh the $BEST$ and $WORST$ outcome and select the alternative with the highest $H = \alpha BEST + (1 - \alpha) WORST$ for some chosen $\alpha \in [0, 1]$. In general, the DM will choose an action a_i with highest 'expected' value $EV_i = F(r_{i,1}, r_{i,2}, ..., r_{i,n})$ where F is some aggregate function whose forms depends on the DM's attitude and the $r_{i,1}$ are the rewards for action a_i for n different states of nature. For decision making under uncertainty, we then obtain an 'expected' value EV_i for each action a_i by using the weights of focal elements in the BoE (instead of probabilities): $E_i = \sum_{B \in \mathfrak{F}_\Theta} r(a_i, B) \cdot m_\Theta(B)$, where $r(a_i, B)$ is the reward for action a_i and state of nature $S \in B$. This leads to a collection of payoffs $R_{a_i,B} = \{EV_{i,j} | S_j \in B\}$ for each a_i and focal element B. The 'expected' value of this collection is then determined using aggregate function F (as described above).

References

[1] U.S. Army, *Field Manual 2-22.3 (FM 34-52)*. 2006.

[2] K. Sambhoos, J. Llinas, and E. Little, "Graphical methods for real-time fusion and estimation with soft message data," in *ICIF*, Germany, 2008.

[3] S. Blackman and R. Popoli, *Design and Analyis of Modern Tracking Systems*. Artech House, 1999.

[4] D. L. Hall and J. Llinas, Eds., *Handbook of Multisensor Data Fusion*. CRC Press, 2001.

[5] A. Benavoli, L. Chisci, A. Farina, and B. Ristic, "Modelling uncertain implication rules in evidence theory," in *ICIF*, Germany, 2008.

[6] D. Lewis, "Probabilities of conditionals and conditional probabilities," *The Phil. Rev.*, vol. LXXXV, pp. 297–315, July 1976.

[7] R. R. Yager, M. Fedrizzi, and J. Kacprzyk, Eds., *Advances in the Dempster-Shafer Theory of Evidence*. Wiley, 1994.

[8] P. Smets, "Practical uses of belief functions," in *UAI*, 1999, pp. 612–621.

[9] G. Shafer, *A Mathematical Theory of Evidence*. Princeton Univ. Press, 1976.

[10] F. Janez and A. Appriou, "Theory of evidence and non-exhaustive frames of discernment: Plausibilities correction methods," *Int. J. of Approx. Reasoning*, vol. 18, pp. 1–19, Jan./Feb. 1998.

[11] E. C. Kulasekere, K. Premaratne, D. A. Dewasurendra, M.-L. Shyu, and P. H. Bauer, "Conditioning and updating evidence," *Int. J. of Approx. Reasoning*, vol. 36, pp. 75–108, Apr. 2004.

[12] K. Premaratne, D. A. Dewasurendra, and P. H. Bauer, "Evidence combination in an environment with heterogeneous sources," *IEEE Trans. Syst., Man, Cyber., A*, vol. 37, pp. 298–309, 2007.

[13] R. Fagin and J. Y. Halpern, "A new approach to updating beliefs," in *UAI*, 1991, pp. 347–374.

[14] ——, "Uncertainty, belief and probability," in *Computational Intelligence*. Morgan Kaufmann, 1989, pp. 1161–1167.

[15] D. A. Dewasurendra, P. H. Bauer, and K. Premaratne, "Evidence filtering," *IEEE Trans. Sig. Proc.*, vol. 55, pp. 5796–5805, 2007.

[16] J. Dzifcak, M. Scheutz, and C. Baral, "What to do and how to do it: Translating natural language directives ...," in *IEEE Int. Conf. Robotics and Auto. (ICRA)*, Japan, May 2009.

[17] R. R. Yager, "OR ordered weighted averaging aggregation operators in muticriteria decisionmaking," *IEEE Trans. Syst., Man, Cyber.*, vol. 18, pp. 183–190, Jan./Feb. 1988.